## Three-Dimensional Geometry

## Fast Track GRASP Math Packet <br> Part 1



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## Three-Dimensional Geometry (Part 1)

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## Three-Dimensional Geometry (Part 1)

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## Overview

| Prerequisites | Before working on this packet, students should complete the Fast Track <br> GRASP Math Packet, Two-Dimensional Geometry, Parts 1 and 2. <br> Students should complete Three-Dimensional Geometry, Part 1 before <br> working on Three-Dimensional Geometry, Part 2. |
| :--- | :--- |
| Connections | Students can find more opportunity to practice with volume and <br> exponents, cubes and cube roots in the Fast Track GRASP Math Packet, <br> The Power of Exponents, Part 1. <br> Students can find more opportunity for practice with the volume <br> formulas in the Fast Track GRASP Math Packet, Tools of Algebra: <br> Equations, Expressions, and Inequalities. |

In this packet, you will explore concepts in three-dimensional (3D) geometry and measurement.

In Part 1, you will study the following topics:

- Types of solids (three-dimensional objects)
- Measuring the surface area of solids
- Measuring the volume of solids
- Geometric formulas

In Part 2. you will build on what you learned in Part 1, and study the following topics:

- Volume in a science context, studying the density of matter
- Measurement conversions

In addition to the learning the topics above, you will find the following materials to help you:

- High School Equivalency Test Practice Questions. You will practice the concepts you have learned from this packet to work on these questions. The answer key for this section explains the correct answers, and also some of the wrong answers.
- A graphic organizer to study vocabulary is included, along with a vocabulary activity to review concepts. A glossary with important terms from this packet is also included for your study.
- Concept Circles can help you make connections between the concepts you have learned and help you remember those connections.


## Assessment Questions

## Calculator allowed

The following questions will help to see if this packet is right for you. Do your best to answer each question. If you can't answer, don't worry-this packet will help you answer questions like these and more. When you are finished with the questions, read our recommendations.

You may refer to the Formulas for Three-Dimensional Geometry or the GED Formula Sheet' ${ }^{1}$.

## Question 1

Which of these solids has the greatest volume? (Note: Figures are not drawn to scale)
A.

C.

B.

D.


## Question 2

Adela wants to paint a cylindrical tank that they bought to store rainwater in their garden. The tank is open at the top and they do not need to paint the bottom of the tank. They only need to paint the outside of the tank. If one gallon can of paint covers 400 square feet, how many cans will they need to buy for the job?

A. 4
B. 5
C. 6
D. 7

[^0]
## Question 3

## Part One

The Quetzal chocolate company sells chocolate bars in triangular prism boxes. The triangular faces are made up of equilateral triangles that measure 60 mm on each side. What is the approximate surface area of the box?

A. $12,480 \mathrm{sq} \mathrm{mm}$
B. $21,120 \mathrm{sq} \mathrm{mm}$
C. $57,120 \mathrm{sq} \mathrm{mm}$
D. $63,360 \mathrm{sq} \mathrm{mm}$

## Part Two

The Quetzal chocolate company sells the boxes shown on the right in packs of 9 boxes. What is the total volume of the 9 boxes of chocolate in this pack?
A. 468,000 cubic mm
B. $2,190,200$ cubic mm
C. $4,212,000$ cubic mm
D. $8,424,000$ cubic mm

(Note: Diagram not drawn to scale)

## Question 4

A wooden cube has an edge length of 6 centimeters and a mass of 137.8 grams. Determine the density of the cube, to the nearest thousandth. Then use the table below to identify the wood.

Which type of wood is the cube made of?
A. Ash
B. Elm
C. Oak

| Type of Wood | Density <br> $\left(\mathrm{g} / \mathrm{cm}^{3}\right)$ |
| :---: | :---: |
| Pine | 0.373 |
| Hemlock | 0.431 |
| Elm | 0.554 |
| Birch | 0.601 |
| Ash | 0.638 |
| Maple | 0.676 |
| Oak | 0.711 |

D. Pine

## Answer Key

Question 1. Choice A.
Question 2. Choice B.
Question 3. Part 1: Choice C. / Part 2: Choice C.
Question 4. Choice A.

## Recommendations

Consider the following when making a decision about working through this packet:

- Student has some difficulty with Question 1, 2, or 3: The student may choose to work through Three-Dimensional Geometry, Part 1.
- Student has some difficulty with Question 4: If a student comfortably answers Questions 1, $2 \& 3$, but has some difficulty with Question 4, the student may feel confident enough to skip Three-Dimensional Geometry, Part 1 and go directly to Three-Dimensional Geometry, Part 2.).
- Student comfortably answers all four questions: The student may choose to work on a different packet. However, it is recommended that students complete the Test Practice Questions in Three-Dimensional Geometry, Part 1 and Three-Dimensional Geometry, Part 2 (Volume \& the Density of Matter), for practice with questions covered in these packets that students are likely to see on the HSE exam.

This assessment asks students to demonstrate understanding of:
Question 1 (from Three-Dimensional Geometry, Part 1): Calculate dimensions, surface area, and volume of three-dimensional figures. (GED Quantitative Problem Solving Assessment Targets Content Indicator: Q.5.a, Q.5.b, Q.5.d, Q.5.e).

Question 2 (from Three-Dimensional Geometry, Part 1): Compute surface area of cylinders. (GED Quantitative Problem Solving Assessment Targets Content Indicators: Q.5.b).

Question 3 (from Three-Dimensional Geometry, Part 1): Compute volume \& surface area of right prisms. (GED Quantitative Problem Solving Assessment Targets Content Indicators: Q.5.c).

Question 4 (from Three-Dimensional Geometry, Part 2): Calculate dimensions, surface area, and volume of three-dimensional figures. (GED Quantitative Problem Solving Assessment Targets Content Indicators: Math - Q.5.a., Physical Science, Chemical Properties and Reactions Related to Living Systems, P.c.2)

## Welcome!

Congratulations on deciding to continue your learning! We are happy to share this study packet on three-dimensional geometry, surface area, and volume. We hope that these materials are helpful in your efforts to earn your high school equivalency diploma. This group of math study packets will cover mathematics topics that we see on high school equivalency exams. If you study these topics carefully, while also practicing other math skills, you will increase your chances of passing the exam.

Please take your time as you go through the packet. You will find plenty of practice here, but it's useful to make extra notes for yourself to help you remember. You will probably want to have a separate notebook where you can recopy problems, write questions, and include information that you want to remember. Writing is thinking and will help you learn.

After each section, you will find an answer key. Try to answer all the questions and then look at the answer key. It's not cheating to look at the answer key, but do your best on your own first. If you find that you got the right answer, congratulations! If you didn't, it's okay. This is how we learn. Look back and try to understand the reason for the answer. Please read the answer key even if you feel confident. We added some extra explanations and examples that may be helpful. If you see a word that you don't understand, try looking at the Vocabulary Review at the end of the packet.

We hope you share what you learn with your friends and family. If you find something interesting here, tell someone about it! If you find a section challenging, look for support. If you are in a class, talk to your teacher and your classmates. If you are studying on your own, talk to people you know or try searching for a phrase online. Your local library should have information about adult education classes or other support. You can also find classes listed here: https://www.acces.nysed.gov/aepp/find-adult-education-program.

You are doing a wonderful thing by investing in your own education right now. You have our utmost respect for continuing to learn as an adult.

Please feel free to contact us with questions or suggestions.
Best of luck!
Mark Trushkowsky (mark.trushkowskyacuny.edu) \& Eric Appleton (eric.appletonacuny.edu) CUNY Adult Literacy and High School Equivalency Program

## Vocabulary

It is important to understand mathematical words when you are learning new topics. These are some of the words you will find in this study packet: cube, cubic inch, depth, prism, solid, surface area, three-dimensional (3D), and volume.

When we learn new vocabulary, it is good to think about your experience with the word.
Asking questions like, "Have I heard this word before?", "When have I heard this word?", "What do I think this word means?" can help you build on what you already know.

Here's how it works. On the next page, you'll find a chart with each of the vocabulary words above. For each word, ask yourself how familiar you are with the word. For example - the word "area." Which of these statements is true for you and your experience with the word "area"?

- I know the word "area" and use it in conversation or writing.
- I know the word "area," but I don't use it.
- I have heard the word "area" but I'm not sure what it means.
- I have never heard the word "area" at all.

In the chart on the next page, read each word and then choose one of the four categories and mark your answer with a $\boldsymbol{V}$ (checkmark). Then write your best guess at the meaning of the word in the right column. If it's easier, you can also just use the word in a sentence.

Here's an example of how the row for "area" might look when you're done:

| Word | I know the <br> word and <br> use the word | I know the <br> word but <br> don't use it | I have heard the <br> word, but l'm <br> not sure what <br> it means | I have never <br> heard the word | My best guess at the <br> meaning of the word <br> (or use the word in a sentence) |
| :--- | :---: | :---: | :---: | :--- | :--- |
| area | $\boldsymbol{V}$ |  |  | A place or location, like a <br> neighborhood or town |  |

This activity is designed to help you start thinking about some of the important words you will find in this packet. As you go through the activities in this packet, you will learn more about these words, what they mean, and how to use them. You will learn more precise definitions that may come up during your high school equivalency exam.

In The Language of Geometry section, you will find a glossary with the definitions of useful vocabulary from the packet and a graphic organizer to help you study them.

\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
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\end{tabular}

## Introduction to Three-Dimensional Geometry

In this packet, we will explore three-dimensional geometry. Three-dimensional geometry is the geometry of three-dimensional space, which is the space we live in.

We use the term "three-dimensional", or 3D, because there are three dimensions: height, width, and depth.

(1D)


TWO-DIMENSIONAL
(2D)

THREE-DIMENSIONAL
(3D)


LENGTH means the measurement of something from one end to the other, the distance between two points. Height, width, depth, base, side, perimeter are all words used to describe length.

In one-dimensional geometry, we generally use the word "length".
In two-dimensional and three-dimensional geometry, we use different words to help us communicate which length we mean, since there is more than one dimension.

In general:

- Width means the distance from side to side.
- Height means the distance from top to bottom.
- Depth means the distance from front to back.

In two-dimensional geometry, we usually use words like "length," "width," and "height."
In three-dimensional geometry we usually use words like "length," "width," "height," and "depth."

## Solids

In three-dimensional geometry, a solid is a three-dimensional object. In this packet we will work with both diagrams of solids and photographs of solids taken in the real world.

In this packet, we will focus on prisms (including rectangular prisms and triangular prisms), cubes, pyramids, cylinders, cones, and spheres. We will also work with composite solids, which are objects made up of more than one type of solid.

## Prisms

Prisms are the first family of solids for you to explore.
Below are some diagrams of a few types of prisms.


Triangular prism


Rectangular prism


Pentagonal prism


Hexagonal prism

There are differences between these objects that we will examine in the next few pages, but let's start with their similarities. How are these objects similar?

Complete the following sentences:

- All of these objects $\qquad$ .
- All of these objects $\qquad$ .
- All of these objects $\qquad$ .
- All of these objects $\qquad$ .

Here are some similarities you might have noticed:

- All prisms have two ends - a top and a bottom. Those two ends are identical in size and shape.
- All prisms have flat faces.
- Prisms have rectangular sides.

Prisms get their names from the shapes at the ends. A prism with a triangle on its two ends is called a triangular prism. A prism with a rectangle on its two ends is called a rectangular prism. A prism with a pentagon on its two ends is called a pentagonal prism. And so on.

A right prism is any prism where one base appears directly above the other base. An oblique prism is a prism that seems like it is tilting. In an oblique prism, the bases do not appear directly above one another when the prism is standing on its base.


Right Prism


Oblique Prism

In this packet, and on the HSE exam, you will be working with right prisms.

## Rectangular Prisms

1. Write at least 4 things you know about rectangular prisms. You can write things you already know or things you notice when you look at these diagrams.


- 
- 
- 

2. What are some examples of rectangular prisms in the world? Where have you seen objects that look like this before?

## Some Helpful Vocabulary When We Talk About Solids

In everyday
language, we might talk about the "sides" of a cube, but this can be confusing. If you hear someone say "sides," you might wonder if they mean the flat part
 of dice where the dots are or if they mean the edge of the flat part where it turns a corner. Using everyday English to describe cubes can be ambiguous, or unclear.

Because the everyday language we use to describe cubes can be confusing, in math we use specific words for these parts of a three-dimensional shape.

The flat part is called a face.
The edge is where different faces of the shape meet. This is the line along the side of the cube.

Finally, the corner point is called a vertex. The vertex is where the different edges of the shape

mathematical language meet. We use the word vertices for more than one vertex.

The words face, edge, and vertex are used for all kinds of 3-dimensional shapes.
3. How many vertices does a rectangular prism have? $\qquad$
4. How many faces does a rectangular prism have? $\qquad$
5. How many edges does a rectangular prism have? $\qquad$

Some examples of rectangular prisms in the world include shipping containers, bricks, a glass aquarium and pizza boxes.


> A brief history of the shipping container ${ }^{2}$
> Manufactured goods have been transported by sea for over 2,000 years. Before 1956, those goods were packaged in different kinds of containers of different sizes - barrels, crates, sacks. It took a lot of work to load and unload those ships. Sometimes dock workers had to unpack a container and repack the goods into another container that could be used to transport the goods by train and by truck. Ships often spent more time at docks than they did sailing over sea.

> That all changed in 1956, when an US truck driver named Malcom McLane introduced the rectangular prism shipping containers that we still use today. These shipping containers revolutionized the shipping industry. Containers could now be loaded from ships directly on to trucks and trains, making shipping a lot more efficient and a lot cheaper. Today 95\% of manufactured goods are transported by container shipping all around the world.

[^1]
## Cube

A cube is a special type of rectangular prism.


What makes a cube so special?
A cube is a rectangular prism where every edge is equal in length.

A cube has 6 identical square faces.


Some examples of cubes in the world include stone blocks, wooden blocks, sugar cubes, dice, and milk crates.

6. What are some other examples of cubes in the world?

## Triangular Prism

Triangular prisms have a triangle face at each end.

7. How many faces do triangular prisms have?
8. How many edges do triangular prisms have?

Here are some examples of triangular prisms in the world.

9. What are some other examples of triangular prisms in the world?

[^2]There are other types of prisms, but we will focus on rectangular prisms, triangular prisms, and cubes because they are the most common and they are the prisms you may see on your HSE exam.

Before we move to another family of solids, let's practice sketching prisms. When you are solving math problems in geometry, it can be helpful to draw quick sketches of the figures you are working with.


## Three-Dimensional Geometry (Part 1)

10. Try sketching your own rectangular prisms, triangular prisms, and cubes.

Use the dotted section if it helps, but make sure to try it in the blank space as well.
Diagrams are helpful problem-solving tools, so it's good to practice sketching them.


## Right Pyramids

The next family of solids we are going to look at are pyramids. Pyramids are named for the shape of their base.

11. A square pyramid has $\qquad$ faces and $\qquad$ edges.
12. A triangular pyramid has $\qquad$ faces and $\qquad$ edges.

A pyramid is a solid object with one base that has faces that are triangles that meet at a point opposite the base called the apex.


The slant height is the distance from the base to the apex, going up the face of the pyramid. Also called slant length. In this diagram, the slant height is labeled $s$.

The height of a pyramid is the distance from the apex to the base. In this diagram, the height is labeled $h$.

Here are some examples of pyramids in the world.

13. What are some other examples of pyramids in the world?
14. Try sketching your own square pyramids below.


## Cylinders

The next solid is called a cylinder.

15. How are cylinders similar to rectangular prisms?
16. How are cylinders different from rectangular prisms?
-
-
$\bullet$
17. What are some examples of cylinders in the world? Where have you seen objects that look like this before?

A cylinder is a solid with a circle at each end. A curved surface connects the two circular faces.

Because the ends are circles, cylinders have a radius ( $r$ ) and a diameter ( $d$ ). The distance around the circular end is a circumference.


Similar to prisms, to sketch a cylinder, draw the two ends and the connect them with two lines.

18. Try sketching a few cylinders below.

## Cone

A cone is a solid with a circular base that is connected to a point by a curved side.

$h$ represents the height of the cone. $r$ represents the radius of the cone. $s$ represents the slant height of the cone.

Cones have things in common with some of the other solids we've looked at so far. Let's review what we know about pyramids and cylinders as we think about cones.
19. A cone is similar to a pyramid because $\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$
20. A cone is similar to a cylinder because $\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$
21. What are some examples of cones in the world? Where have you seen objects that look like this before?

## Sphere

A sphere is a three-dimensional object shaped like a ball. Every point on the surface is the same distance from the center of the sphere. That distance is called a radius $(r)$ of the sphere. The straight distance from one point on the surface, through the center, to another point on the surface is the diameter (d) of the sphere. As in a circle, the diameter of a sphere is 2 times the distance of the radius. Or put another way, the radius is half the distance of the diameter.



A hemisphere is exactly $1 / 2$ of a sphere.

Cherry tomatoes, dorodango ${ }^{4}$, bowling balls, and bubbles are all examples of
 spheres in the world. Even a spinning coin can create the image of a sphere!


[^3]22. What are some other examples of spheres in the world?

To sketch a cone, draw an oval. Add an apex and then connect the apex to the circular base.

To sketch a sphere, draw an circle. Then draw an oval in the middle of the circle.
 ircle.

23. Try sketching a few cones and spheres.
24. The world is full of three-dimensional solids. Practice identifying solids by writing the name of any solids you see in each picture.


The courtyard of the Louvre Museum in Paris, France


Storm King Art Center, New Windsor NY


25. Complete the table comparing 3D solids. (Note: There is more than one correct answer.)


| 3D Solid | Which two solids do you <br> think are similar to this solid? |  | Explain your reasoning. (You may use <br> the diagrams above or draw your own.) |
| :--- | :--- | :--- | :--- |
| Rectangular <br> Prism |  |  |  |
| Square <br> Pyramid |  |  |  |
| Cylinder |  |  |  |
| Cone |  |  |  |
| Sphere |  |  |  |
| Cube |  |  |  |

## Introduction to Three-Dimensional Geometry - Answer Key

1. There are no wrong answers to this question.
2. Some examples of rectangular prisms in the world include cereal boxes, shipping boxes, tables, books, refrigerators, microwaves, bathtubs, bookcases, apartment buildings, iPhones, bars of soap, etc.
3. A rectangular prism has 8 vertices.
4. A rectangular prism has 6 faces.
5. A rectangular prism has 12 edges,
6. Some examples of cubes in the world include tissue boxes, Rubix cubes, etc.
7. Triangular prisms have 5 faces.
8. Triangular prisms have 9 edges.
9. Some examples of triangular prisms in the world include some rooftops, tents, cheese wedges, etc
10. Sketches of rectangular prisms, triangular prisms, and cubes.
11. A square pyramid has 5 faces and 8 edges.
12. A triangular pyramid has 4 faces and 6 edges.
13. Some examples of pyramids in the world include some rooftops,
14. Sketches of pyramids.
15. Some possible answers include: they both have a top and a bottom that are identical. They both have height. You can find examples of both in the supermarket.
16. Some possible answers include cylinders roll and prisms do not. Cylinders have a curved side and prisms only have straight edges. Rectangular prisms have 6 faces, cylinders only have 2.
17. Some examples of cylinders in the world include cans, piping, some containers, water tanks, cigarettes, etc.
18. Sketches of cylinders.
19. Possible answers include: Both a cone and a pyramid have a point (an apex). Both a cone and a pyramid have a height and a slant height.
20. Possible answers include: Both have a circular base. Both have a radius, diameter, and circumference. Both have a curved side. Both would roll.
21. Some examples of cones in the world include party hats, megaphones, some rooftops, some paper cups, ice cream comes, traffic cones, etc.
22. Some examples of spheres in the world include marbles, balls, beads, pearls, etc.
23. Sketches of cones and spheres. .
24. Grain silo - cylinder and cone. Museum courtyard - square pyramid. Party hat - cone. Storm King - rectangular prisms. Tent -triangular prism. Basketballs - spheres. Tin can - cylinder. Logs - cylinders. Shed - rectangular prism and square pyramid. Sewer pipes - cylinder. Shipping container - rectangular prism. Sculpture - cube
25. There are multiple correct answers here. The point is to explain your reasoning. Here are some possible responses.

- A rectangular prism is similar to a cylinder and square pyramid. It is similar to a cylinder because they both have a top and bottom that are the same. A rectangular prism is similar to a square pyramid because they both have a rectangular base and height.
- A square pyramid is similar to a cone and cube. A pyramid is similar to a cone because they both have one flat base and a point at the top. A square pyramid is similar to a cube because they both have a square base.
- A cylinder is similar to a cone and sphere. A cylinder is similar to a cone because they both have a circular base. A cylinder is similar to a sphere because they will both roll.
- A cone is similar to a cylinder and square pyramid. A cone is similar to a cylinder because they both have circular bases and height. A cone is similar to a pyramid because they both have an apex and a slant height.
- A sphere is similar to a cone and cube. A sphere is similar to a cone because they both have a circle at their widest part. A sphere is similar to a cube because in both solids, the distance from one side to the other is the same distance as from the top to the bottom.
- A cube is similar to a rectangular prism and a sphere. A cube and a rectangular prism both have 6 faces and 12 edges. If a cube had an edge length that was equal to the diameter of a sphere, then the sphere would fit perfectly in the cube.


## Surface Area of Prisms and Pyramids

We use the word area to describe the size of the surface of a two-dimensional shape. We use surface area to describe the size of the surface of a three-dimensional object. Surface area is like covering a gift box with wrapping paper or wrapping a potato in aluminum foil.


In 2021, the artists Christo and Jeanne-Claude covered the surface area of the Arc de Triomphe in Paris, France with over 25,000 square meters of fabric.

Our strategy for calculating the surface area of any three-dimensional solid is to calculate the area of each surface. Just like area in two-dimensional geometry, we measure surface area based on how many squares it takes to completely cover a surface.

Consider this rectangular prism.


1. How many squares do you think it takes to cover the entire surface of this prism?
2. Describe how you figured out how many squares it would take to cover the prism.
[^4]A rectangular prism has six faces and in this diagram we can see three of them. The three sides we can see help us figure out the surface area of the sides we can't see.


We can see 12 squares covering the left side of the prism and we can imagine 12 squares covering the right side of the prism.

We can see 18 squares covering the top side of the prism and we can imagine 18 squares covering the bottom of the prism.

We can see 6 squares covering the front of the prism and we can imagine 6 squares covering the back face of the prism.
3. In question 1, you figured out how many squares it would take to cover the surface of this rectangular prism. Find a different way to add up the total number of squares.

Find the surface area of these three-dimensional figures.
4.
_______
squares

5.

6. $\qquad$ squares

7. ___ squares


We can use geometry nets to help us keep track of all of the sides and surfaces of 3D objects. Nets are 2D models that can be folded into 3D objects. Nets can help us understand and determine surface area.

For example, here is a net for the rectangular prism you looked at in the beginning of this session. Can you imagine folding up the net to make the rectangular prism?



## Surface Area of Rectangular Prisms

8. Find the area of Net 1.

9. In a few sentences, describe how you calculated the area of Net 1.
10. Calculate the area of Net 2.

11. In a few sentences, describe how you calculated the area of Net 2.
12. Calculate the area of Net 3.

13. In a few sentences, describe how you calculated the area of Net 3.
14. What ideas do you have for calculating the surface area of any rectangular prism?

15. Which Net (1, 2, or 3) matches Rectangular Prism A? How do you know?
16. Which Net (1, 2, or 3) matches Rectangular Prism B? How do you know?
17. Between Prism A and Prism B, which Rectangular Prism has a larger surface area?

You have developed your own strategies for calculating the surface area of rectangular prisms.

One of the powerful things about math is that there are always different ways to make sense of patterns and relationships. That means there is always more than one way to approach problems.

The Math Formula Sheet for the HSE Exam lists the formula for calculating the surface area of a rectangular prism as $S A=2 l w+2 l h+2 w h$.

The Math Formula Sheet for the HSE Exam also lists the formula for calculating the surface area of a right prism as $S A=p h+2 B$, where $p$ is the perimeter of the base, $h$ is the height of the prism, and $B$ is the area of the base.

Rectangular prisms are right prisms. Any formula that works for right prisms will work for any rectangular prism. It is strange that the Math Formula Sheet has two formulas.

Let's see how both formulas work.

Let's imagine we need to calculate the surface area of a stone block with the measurements in the diagram.


| $S A$ of rectangular prism $=2 l w+2 l h+2 w h$ | SA of right prism $=p h+2 B$ |
| :---: | :---: |
| $S A=2(5)(3)+2(5)(2)+2(3)(2)$ | $p=3+5+3+5$ |
| $S A=30+20+12$ | $p=16$ |
| $S A=62$ square feet | $B=3 \times 5$ so $B=15$ |
| $S A=16(2)+2(15)$ |  |
| $32+30=62$ square feet |  |

Calculate the surface area of the following rectangular prisms, using any method you choose. (Note: Use the space below for your work. It may be helpful to make sketches to help you keep track of the measurements.)
18. What is the surface area of a rectangular prism that is 6 inches by 4 inches by 8 inches?

$$
\text { Surface Area }=\ldots \ldots \text { ___ square inches }
$$

19. What is the surface area of a rectangular prism that is 14 inches by 3 inches by 7 inches?
Surface Area = _____ square inches
20. What is the surface area of a rectangular prism that is 12 inches by 15 inches by 15 inches?

Surface Area = $\qquad$ square inches
21. What is the surface area of a rectangular prism that is 12 inches by 12 inches by 18 inches?

Surface Area = $\qquad$
22. What is the surface area of a rectangular prism that is 40 feet by 8 feet by 10 feet?

Surface Area = $\qquad$
23. Find the possible dimensions of a rectangular prism that has a surface area of 72 square inches. (Note: There is more than one possible answer.)

## Surface Area of Other Right Prisms

To calculate the surface area of a triangular prism, we need to calculate the area of each of the five faces.


Let's start with the two triangular bases. The formula for calculating the area of a triangle is $A=1 / 2 b h$, where $\boldsymbol{b}$ is the length of the base and $\boldsymbol{h}$ is the height of the triangle.
24. What is the area of the triangular base?
25. What is the total area of both triangular bases?

In order to calculate the area of the rectangular faces, it can be helpful to make a net.
26. Use the diagram above to fill in the missing measurements in the net below.

27. Now that you have labeled the measurements of each side of the rectangle, find the area of each rectangle.

28. Now that you have the area of each face of the prism, add them all together to find the total surface area.

12 sq. ft. $\quad+$ $\qquad$ sq. ft. + $\qquad$ sq. ft. $\quad+$ $\qquad$ sq. ft. $+$ $\qquad$ sq. ft. = area of triangle area of triangle area of rectangle area of rectangle area of rectangle
$\qquad$ sq. ft.
surface area of right prism with triangular base


Take another look at the 3 rectangular faces of the triangular prism you've been working on.


Notice what happens when we push the rectangles together:

$11 \times(5+6+5)$
$11 \times 16=176$

We can combine the areas of the three rectangles into one large rectangle with the same area.

And what do we know about the three large rectangle?

- We know that one side length is equal to the height of the prism.
- We also know that the other side length is equal to the perimeter of the triangle face.

How do we know that?

Remember the net we made from this triangular prism. The side lengths of the triangle became the side lengths of the rectangles.


So to find the surface area of all of the rectangular faces of the prism, we can multiply the perimeter of the base times the height of the prism. Then we add the area of the 2 triangular bases. To summarize that, we can use the formula:

29. Calculate the surface area of this triangular prism. If it is helpful, you can use the net provided

A. 2,066 square feet
B. 2,160 square feet
C. 2,336 square feet
D. 2,400 square feet
30. Calculate the surface area of this triangular prism. Note: No net is provided this time, so you can practice sketching one if it is helpful.


## Surface Area of Pyramids

Let's start with a pyramid with a square base.
Imagine "opening" a square pyramid and converting it into a net.


How would you figure out the surface area of the pyramid?

When we "open" up a square pyramid, we end up with a square and four triangles. We can use what we know about calculating the area of squares and triangles to calculate the surface area of the pyramid.

The base of the pyramid is a square. We can calculate the area of a square by multiplying two of its sides together.

The other faces of the pyramid are triangles. We can calculate the area of a triangle with the formula, $\mathbf{A}=1 / 2 \mathrm{bh}$.

In order to calculate the surface area of a pyramid, we need to know the height of the triangular faces.

## The slant length of the pyramid is the height of the triangular faces.

If we open up the pyramid, we can see the slant length as the height of the triangular faces.

31. What is the surface area of the pyramid above?
32. What is the surface area of this pyramid?
A. 392 square cm

B. 544 square cm
C. 800 square cm
D. 2,368 square cm
33. What is the surface area of a square pyramid with a base length of 40 mm and a slant length of 29 mm ? (It might help to draw a diagram to help you keep track of the different parts of the pyramid.)
$\qquad$ sq. mm.
34. The square pyramid in the diagram below has a surface area of 360 sq ft . What is the slant length of the pyramid?

A. 13 ft
B. 14 ft
C. 15 ft
D. 16 ft

There are other methods we could use to calculate the surface area of a pyramid.
The one that is printed on the HSE Formula Sheet is:
$S A=1 / 2 p s+B$, where $p$ is the perimeter of the base, $B$ is the area of the base, and $s$ is the slant length.


This pyramid has a base length of 6 m and a slant length of 5 m .

Method 1 is what we have been doing so far.
Method 2.

| SA = Area of Base + (1/2 bh $\times 4)$ | $S A=1 / 2 p s+B$ |
| :---: | :---: |
| Step 1: Calculate the area of the base. | Step 1: Calculate the perimeter of the base. |
| 6 meters $\times 6$ meters $=36$ sq meters | $6 m+6 m+6 m+6 m=24 m$ |
| Step 2: <br> Calculate the area of one triangular face. | Step 2: Calculate the total area of all four triangular faces. Use the formula $1 / 2 p h$ where $p$ is the perimeter of the base. |
| $1 / 2(6 \mathrm{~m} \times 5 \mathrm{~m})=15 \mathrm{sq} \mathrm{m}$ | $1 / 2(24$ meters $\times 5$ meters $)=60$ meters |
| Step 3: <br> Multiply the area of one triangle by 4 | Step 3: <br> Calculate the area of the base, $B$ |
| $15 \mathrm{sq} \mathrm{m} \times 4=60$ meters | 6 meters $\times 6$ meters $=36$ sq meters |
| Step 4: Add the area of the base to the total area of the triangular faces. | Step 4: Add the area of the triangular faces to the area of the base. |
| 36 sq m +60 sq m $=96$ sq m | 60 sq m +36 sq $m=96$ sq m |

These methods will always produce the same answer and they will both always work.
Use whichever one makes sense to you.

## Surface Area of Prisms and Pyramids - Answer Key

1. 72 squares
2. There are many effective strategies for calculating the area of the net.
3. There are many effective strategies for calculating the area of the net.
4. $\quad 142$ squares
5. 84 squares
6. 68 squares
7. 96 squares
8. 174 square units
9. There are many effective strategies for calculating the area of the net.
10. 176 square units
11. There are many effective strategies for calculating the area of the net.
12. 166 square units
13. There are many effective strategies for calculating the area of the net.
14. There are many patterns you might have noticed to help you as you calculated the areas of each net.
15. Net 3
16. Net 1
17. Net 1
18. $\quad 208 \mathrm{sq}$ in
19. $\quad 322 \mathrm{sq}$ in
20. 1170 sq in
21. 1152 sq in
22. $1,600 \mathrm{sq} \mathrm{ft}$
23. One possible solution is 2 inches by 3 inches by 6 inches. Another possible solution is 2 inches by 2 inches by 8 inches.
24. 12 square feet
25. 24 square feet
26. 


27. $5 \times 11=55$ square feet, $6 \times 11=66$ square feet, $5 \times 11=55$ square feet
28. $24+110+66=200$ square feet
29. Choice B.

30. 192 square meters.
31. $384 \mathrm{ft}^{2}$. The area of the base is $144 \mathrm{ft}^{2}$. The area of each triangle is $60 \mathrm{ft}^{2}$. The total area of all 4 triangles is $240 \mathrm{ft}^{2} .144 \mathrm{ft}^{2}+240 \mathrm{ft}^{2}=384 \mathrm{ft}^{2}$.
32. Choice $\mathbf{C}$. The area of the square base is 256 sq cm . The area of each triangle is 136 sq cm . The total area of all 4 triangles is $544 \mathrm{sq} \mathrm{cm} .256 \mathrm{sq} \mathrm{cm}+544 \mathrm{sq} \mathrm{cm}=800 \mathrm{sq} \mathrm{cm}$.
33. $3,920 \mathrm{sq} \mathrm{mm}$. The area of the square base is 1600 sq mm . The area of each triangular face is 580 sq mm . The total area of all 4 triangular faces is $2,320 \mathrm{sq} \mathrm{mm} .1600 \mathrm{sq}$ $\mathrm{mm}+2,320 \mathrm{sq} \mathrm{mm}=3,920 \mathrm{sq} \mathrm{mm}$
34. Choice A. You can try each of the possible answers and use trial and error to solve this problem. The surface area of the base is 100 square ft . If the slant length is 13 , the area of each triangular face would be 65 sq feet, and the total area of all 4 triangular faces would be 260 sq ft. Since $100+260=360$, then we know 13 is the slant length for this pyramid.

## Surface Area of Round Objects

## Surface Area of Cylinders

Examine these photographs of the surface area of a can and its label.


What do you notice?
What do you wonder?

One thing you may have noticed is that we can "open" the surface area of a cylinder to make two circles and a rectangle.

Cylinders have identical circles at the top and a circle on the bottom and a curved part that opens up into a rectangle. We can calculate the surface area of a cylinder by calculating the area of the two circles and the area of the rectangle.


1. Use the squares to estimate the surface area of the cylinder below.


The formula for the area of a circle is $\mathbf{A}=\pi \mathbf{r}^{2}$ (using 3.14 as an estimate for $\pi$ )

2. If the radius of the cylinder in the diagram above is 3 inches, calculate the area of each circle.
3. What information do you need to calculate the area of the rectangle?


As we have seen, the curved part of a cylinder opens into a rectangle. By cutting open a cardboard tube, we can learn more about the surface area of the curved part of a cylinder.

4. What do you notice about the relationship between the cylinder and the rectangle?

One thing you may have noticed is that the height of the cylinder is equal to the length of one side of the rectangle. If you didn't notice that, look back at the dotted black line in the photos of the cardboard tube.

Another thing you may have noticed is that the distance around the circular ends of the cylinder is equal to the length of the other side of the rectangle. If you didn't notice that, look back at the dotted white line in the photos of the cardboard tube.

The distance around a circle is called the circumference. The circumference of the cylinder is the same length as the side of the rectangle.

The formula for the circumference of a circle is $\mathbf{C = 2 \pi r}$ or $\mathbf{C}=\boldsymbol{\pi d}$ (and we use 3.14 as an estimate for $\pi$ )


When calculating the surface area of a cylinder, we need at least two pieces of information.

- either the radius or diameter of the cylinder
- the height of the cylinder

5. Calculate the surface area of the cylinder shown in the diagram below.


The HSE Formula Sheet gives the formula for calculating the surface area of a cylinder as:

$$
\mathrm{SA}=2 \pi r h+2 \pi r^{2}
$$

We can see two parts of the surface area in the formula: $\mathbf{2 \pi r h}$ and $\mathbf{2 \pi} \boldsymbol{r}^{2}$. It is important to understand what each part of the formula represents.
6. Which part of the formula tells us the surface area of the circular bases of the cylinder, $2 \pi r h$ or $2 \pi r^{2} ?$
Explain your thinking.
7. Which part of the formula tells us the surface area of the curved part of the cylinder, $2 \pi r h$ or $2 \pi r^{2} ?$
Explain your thinking.

As it is given on the HSE Formula Sheet, the formula is for calculating the surface area of a cylinder with a top and bottom. The reason why it is important to understand the different parts of the formula is that not all cylinders are the same.

8. What differences do you see in the surface area of these 3 cylinders?

Not all cylinders are the same. Some have a top and bottom, like the can of tomatoes. Some don't have a top or a bottom, like the piece of metal piping. Some only have a bottom, like the plastic container.
9. Below are 6 different cylinders. Some have a top and a bottom. Some don't have a top or a bottom. Some have only a bottom.

Write the formula, or part of the formula, you would use to calculate the surface area of each one in the space below the cylinder.
$2 \pi r h+2 \pi r^{2}$
2mrh
$2 \pi r h+\pi r^{2}$
$2 \pi r^{2}$

| Unopened can of tomatoes | A trash can with no lid | A protective tree covering |
| :---: | :---: | :---: |
|  |  |  |
| MUIR GLEN |  |  |
|  |  |  |
| Metal piping |  |  |

## Practice

10. Fusani is buying plastic piping for a project.

The pieces he wants to buy are 14 inches long and have a diameter of 4 inches.
What is the surface area of each piece of pipe? ( $\pi \sim 3.14$ )
A. 25.12 square inches
B. 175.84 square inches
C. 188.4 square inches

D. 200.96 square inches
11. A cardboard tube has a diameter of 45 mm and a height of 100 mm .

What is the surface area of the tube? ( $\pi \approx 3.14$ )
A. $14,130 \mathrm{sq} \mathrm{mm}$
B. $15,719.625 \mathrm{sq} \mathrm{mm}$
C. $17,309.25 \mathrm{sq} \mathrm{mm}$
D. $26,847 \mathrm{sq} \mathrm{mm}$

12. This cylindrical Bluetooth speaker has a height of 19 cm and a radius of 4 cm . What is the surface area of the speaker, including the top and bottom? ( $\pi \approx 3.14$ )
A. $100.48 \mathrm{~cm}^{2}$
B. $477.28 \mathrm{~cm}^{2}$
C. $577.76 \mathrm{~cm}^{2}$
D. $1,356.48 \mathrm{~cm}^{2}$


## Surface Area of Cones

Imagine cutting a cone from the tip (also called an apex) and the base. This part of the cone is called the slant length. It is usually represented with the letter $s$ or the letter $l$.


When we flatten out our cone, we see two sections.

- the flat circular base of the cone
- the curved surface area of the cone

The flat circular base of the cone
We can use the formula for the area of a circle to find surface area of this part of the cone. $\mathrm{A}=\pi \boldsymbol{r}^{2}$

The circumference of the circular base is outlined with a dotted white line. It is the same length as the dotted white line along the curved surface of the cone.

The curved surface area of the cone The white dotted line is equal to the circumference of the base of the cone. The other two sides are equal to the slant length.


Let's focus on the area of the curved surface of the cone.

Again, the dotted white line represents the circumference of the circular base (or $2 \pi r$ ).

Imagine cutting the curved surface of the cone into long triangles. For this cone, we have cut the curved section into 17 triangles.

The long sides of all of the triangles would be equal to the slant length (s).


We can arrange the triangles into a rectangle.

What do you notice?

What do you wonder?


One side of the rectangle is equal to the slant length of the cone.

But what about the dotted white line?

Since opposite sides of a rectangle are equal, we know that the top and the bottom will be the same length.
We also know that the entire dotted white line is equal to $2 \pi r$.

So each side of this rectangle will be $\pi r$ in length.

To find the area of a rectangle, we multiply the length times the width.
The area of this rectangle is $\pi r$ times $s$ or $\pi r s$. And this rectangle has the same area as the curved section of the cone we cut up to build it.

The formula for finding the surface area of the curved part of a cone is $\pi r s$.


Similar to cylinders, we need to understand the situation to make the right choice about which part of the formula for the surface area of a cone to use.
13. Below are 6 different cones. Some have a circular bottom surface and some do not.

Write the formula, or part of the formula, you would use to calculate the surface area of each one in the space below each cone.
$\pi r s+\pi r^{2}$
$\pi r s$
$\pi r^{2}$

| An ice cream cone | A rice hat | A beeswax candle |
| :---: | :---: | :---: |
|  |  |  |
|  |  |  |
| Conical roof | Solid conical game pieces | Conical lamp shades |
|  |  |  |
|  |  |  |

14. This conical roof has a radius of 20 feet and a slant height of 35 feet. What is the surface area of the roof of this building? ( $\pi \approx 3.14$ )
A. 2,198 square feet
B. 3,454 square feet
C. 4,396 square feet
D. 9,420 square feet

15. A craft store is selling large, solid wooden cones. The cones have a slant length of 19 inches and a radius of 6 inches. What is the surface area of each cone? ( $\pi \approx 3.14$ )

16. A cone with no base has a surface area of 816.4 square meters and a radius of 13 meters. What is the length of the slant length of the cone?

## Surface Area of Spheres

Examine these 4 world maps.


Mercator Projection (1569)


Peters Equal-Area Projection (1973)

What do you notice?


Boggs Projection (1929)


Strebe Projection (1995)
What do you wonder?

These 4 maps are only 4 of the hundreds of different world maps that have been created over the past 500 years. The question of how to represent the surface area of a sphere on a flat map is not an easy one to answer. Map makers have been working on that question for centuries. When we try to flatten a sphere, the sphere gets distorted. All flat world maps can represent important information, like shape, distance, land area, and direction. But no flat world map can represent all of those things perfectly. Map makers have to make choices about what information their map will focus on.

Because it is so difficult to flatten the surface area of a sphere, we cannot make a net for a sphere. But there are other things we can do to understand how to measure the surface area of a sphere.

To better understand the surface area of spheres, we are going to do an experiment. We are going to focus our experiment on a tennis ball, but any spherical object will work. An orange or grapefruit will work particularly well.

Step 1:
Cut a sphere (tennis ball, orange, grapefruit, etc) right down the middle into two hemispheres.


Step 2:
Use those hemispheres to trace 4 circles.


What do those circles have in common with the sphere?

One thing the sphere and the circles have in common is that they have the same radius.


Step 3: Start cutting up the tennis ball into small pieces. Use those pieces to cover the area of each of the circles you drew.


If we keep cutting until we have no more tennis ball, this is what we will get.

What do you notice?

When we cut the surface area of a sphere into small pieces, those pieces will cover the area of 4 circles drawn with the same radius as the sphere.


We can calculate the area of a circle using the formula $\mathbf{A}=\boldsymbol{\pi} \boldsymbol{r}^{2}$.
Since we can see that the surface area of a sphere is the same as 4 of the circles...

17. A garden store is selling cement spheres that have a diameter of 36 inches. What is the surface area of one of the spheres?
A. 452.16 square inches
B. $1,017.36$ square inches
C. 4069.44 square inches
D. $16,277.76$ square inches
18. If a basketball has a radius of 12 cm , what is the surface area of the basketball?

19. The moon has a radius of about 1,080 miles. What is the approximate surface area of the moon?

20. The distance from the surface to the Earth to the center of the Earth is about 3,959 miles. What is the approximate surface area of the Earth?


## Surface Area of Round Objects - Answer Key

1. Because it's an estimate, there are many reasonable and correct answers, but your estimate should be between 158 squares and 197 squares.
2. The area of each circle is $\mathbf{2 8 . 2 6}$ square inches. The total area of both circles is 56.52 square inches.
3. We can calculate the area of a rectangle if we know the lengths of its sides - length and width
4. There are many things you might notice. Here are just a few things you may have noted: This is a cardboard tube that has been cut along its height, laid on its side, and flattened into a rectangle. Each picture has dotted white lines and dotted black lines. Sometimes the dotted white lines are in the shape of a circle and sometimes they are in the shape of a line.
5. The surface area of the cylinder is 276.32 square feet.
6. $2 \pi r^{2} . \pi r^{2}$ is the area of a circle and $2 \pi r^{2}$ is the area of 2 circles - the top and the bottom of the cylinder.
7. $2 \pi r h$ is the area of the curved surface of the cylinder. If we "unroll" the curved part of a cylinder, we get a rectangle. $2 \pi r$ is the length of one side of the rectangle and $h$ is the length of the adjacent side. The area of that rectangle can be expressed as $2 \pi r x h$ or $2 \pi r h$.
8. There are many differences you might have noticed. The three cylinders are made of different materials - metal and plastic. You can see through some of them and not others. Some of them have a top and bottom, some have just a bottom, some have no top or bottom. One has something in it, one is empty and one cannot hold anything.
9. 

| tomatoes $2 \pi r h+2 \pi r^{2}$ | trash can $2 \pi r h+\boldsymbol{\pi} r^{2}$ | tree protector $2 \pi r h$ |
| :--- | :--- | :--- |
| metal piping $2 \pi r \boldsymbol{h}$ | container $2 \pi r h+\pi r^{2}$ | glass vase $2 \pi r h+\pi r^{2}$ |

10. Choice B.
11. Choice A.
12. Choice C.
13. 

| Ice cream $\boldsymbol{\pi r s}$ | Rice hat $\boldsymbol{\pi r s}$ | Beeswax candle $\boldsymbol{\pi r s}+\boldsymbol{\pi} \mathbf{r}^{2}$ |
| :--- | :--- | :--- |
| roof $\boldsymbol{\pi r s}$ | Game pieces $\boldsymbol{\pi r s}+\boldsymbol{\pi} \mathbf{r}^{2}$ | lamp shades $\boldsymbol{\pi r s}$ |

14. Choice A. Since there is no base surface area to this roof, we can focus on the curved section of the cone and use the $\pi r s$ part of the formula.
$(3.14) \times(20) \times(35)=2198$ square feet.
15. 357.96 square inches
16. The formula for the surface area of a cone is $S A=\pi r s+\pi r^{2}$. Since there is no base to this cone, we only need to use $\mathrm{SA}=\pi r s$.

| s | r | rrs | SA |
| :---: | :---: | :---: | :---: |
| 20 | 13 | 816.4 | 816.4 |

17. Choice C. 4069.44 square inches
18. 452.16 square centimeters.
19. The surface area of the moon is approximately $14,649,984$ square miles
20. The surface area of the Earth is approximately $196,861,433.36$ square miles

## Volume of Prisms and Pyramids



Volume is the measurement of how much space there is inside an 3D object.

One way to think about how much space there is inside an object is to think about how much it can hold.

For example, the volume of this glass is how much space there is inside the glass to hold water.

How much space there is inside an object can also mean how much space it takes up.

You can't hold water in the space underneath this small table, but this table does have volume.

When we talk about volume, we are talking about the measurement of how much space there is inside the object.


When we measure area and surface area, we are measuring how many squares it takes to cover a surface. We measure volume based on how many cubes it takes to completely fill the space inside an object.


The 36 stands for the number of cubes it takes to fill the space of this rectangular prism.
The word cubic is used because we are filling the entire prism with cubes.
We use cubic inches, because each cube measures 1 inch on each side.


A square inch is a square that measures 1 inch on each side, length and width.
A cubic inch is a cube that measures 1 inch on each edge, length, width, and height.
It takes 36 of these cubic inches to make up the volume of this rectangular prism.
There are a few ways we can write 36 cubic inches:
36 cubic inches
36 cu. in.
36 in $^{3}$

Any measurement in volume will have a number, a unit (in this example the unit is "cubic inches"). Each part of that measurement means something.


1. Fill in the missing boxes in this table.

2. Which of these are other ways to write one cubic inch?
A. 1 in
B. $1 \mathrm{in}^{2}$
C. $1 \mathrm{in}^{3}$
D. $1 \mathrm{cu} . \mathrm{in}$.
3. Which of these are other ways to write one cubic centimeter?
A. 1 cm
B. $1 \mathrm{~cm}^{2}$
C. $1 \mathrm{~cm}^{3}$
D. $1 \mathrm{cu} . \mathrm{cm}$.
E. 1 cc

## ESTIMATING WITH COMMON UNITS OF VOLUME

When we measure volume, our units must have volume too. Remember, finding the volume of an object is asking how many cubes it would take to fill the space of the object. To measure volume, we use cubic centimeters, cubic inches, cubic feet, cubic yards, etc. To help us get an idea of what these units mean, let's estimate with some everyday items.

A cubic centimeter is the volume equal to a cube that measures 1 centimeter on each side.

- A teaspoon has a volume of approximately 1 cubic centimeter
- A 6-sided die has a volume of about 4 cubic cm

4. What is something with a volume of about 25 cubic centimeters?
5. What is something with a volume larger than 500 cubic centimeters?

A cubic inch is the volume equal to a cube that measures 1 inch on each side.

- A tablespoon is approximately 1 cubic inch
- A marshmallow is approximately 2 cubic inches

6. What is something with a volume of about 20 cubic inches?
7. What is something with a volume of more than 1,000 cubic inches?

A cubic foot is the volume equal to a cube that measures 1 foot on each side.

- A plastic milk crate has the volume of approximately 1 cubic foot
- A stack of 45 vinyl records has a volume of approximately 1 cubic foot

8. What is something with a volume of about 5 cubic feet?
9. What is something with a volume of about 25 cubic feet?
10. What is something with a volume of about 1,000 cubic feet?

A cubic yard is the volume equal to a cube that measures 1 yard on each side.


- The volume of 9 wheelbarrows is approximately 1 cubic yard.
- The stack of boxes in this picture has a volume of approximately 1 cubic yard.

11. What is something with a volume of about 2 cubic yards?
12. What is something with a volume of about 3,000 cubic yards?

## Volume of Rectangular Prisms

We can use actual cubes to understand how to calculate volume. Then we'll look at how to calculate volume with only measurements.


I want to know the volume of this cardboard box.

I placed cubic inches (made of wood) along the sides at the bottom of the box.

Each cube is 1 inch in length, width and height.

12 cubes fit along the length of the box. 6 cubes fit along the width of the box.
13. What is the length of the box (in inches)?
14. What is the width of the box (in inches)?
15. How many cubes would it take to cover the base of the box?


72 cubes can fit in the space along the bottom of the box.

What more information would I need to know if I wanted to figure out how many cubes it would take to fill the entire box?
16. How many cubes do you think it will take to fill all of the space inside the box? In a few sentences, describe how you calculated your answer.


72 cubes can fit in the space over the bottom of the box.

6 cubes can fit along the height of the box.

One way you might have calculated your answer is:


72 cubes fit in the space along the bottom of the box.

If 6 cubes can fit along the height, then there can be 6 layers with 72 cubes in each layer.
$72+72+72+72+72+72=432$
or
$72 \times 6=432$.
So 432 cubic inches can fit in the space inside this box!

Let's summarize our steps for calculating the volume of the box.

First we found the area of the base of the box by multiplying the length by the width.

12 inches $\times 6$ inches $=72$ square inches
Then we multiplied the area of the base of the box by the height.

72 square inches $\times 6$ inches $=432$ cubic inches


6 in

Let's practice with volume and rectangular prisms.
17. Directions: Using the digits 1 through 9 at most one time each, fill in the boxes to create a rectangular prism with a volume that is less than 100 cubic units. How many possible answers can you calculate? ${ }^{6}$

18. Directions: Using the digits 1 through 9 at most one time each, fill in the boxes to create a rectangular prism with a volume that is greater than 100 cubic units. ${ }^{7}$


[^5]19. Alex needs to design a box in the shape of a rectangular prism with a volume of 48 cubic inches. Try to find at least 2 different possibilities for the dimensions of the box.


To help you think about this problem, you might imagine 48 cubic inches. How could you arrange them into a rectangular prism?
20. Find the surface area of the rectangular prisms you came up with in the last question.
21. Alex is designing another box in the shape of a rectangular prism. This time, they need a box with a volume of 64 cubic inches. In order to cut down on the cost of materials, Alex needs the box to have a surface area less than 100 square inches. What is a possible length, width, and height for the box?

Questions 22-24
Liv's family built 4 rectangular raised beds to grow vegetables and flowers. They need to calculate the total volume of the raised beds so that they can order soil to fill the beds.


## Each bed has a height of 1.5 feet.

- Bed $A$ is 4 feet by 4 feet
- Bed B is 12 feet by 4 feet
- Bed C is 10 feet by 4 feet
- Bed D is 4 feet by 4 feet

22. How many cubic feet of soil does Liv's family need to fill all 4 raised beds?

The garden supplier where Liv's family wants to order the soil from sells their soil by the cubic yard. So Liv's family needs to convert from cubic feet to cubic yards.

One yard equals 3 feet.
Below is 1 cubic yard. It measures 1 yard by 1 yard by 1 yard. It also measures 3 feet by 3 feet by 3 feet.
23. How many cubic feet are there in 1 cubic yard?

24. How many cubic yards of soil does Liv's family need to fill their 4 raised beds? (Note that the garden supplier can only deliver whole number amounts of cubic yards)
25. A construction company was hired to build a platform. Based on the plans below, what will the volume of the platform be?

26. The length and width of a rectangular prism are 10 feet and 12 feet. If the volume of the prism is 1,440 cubic feet, what is the height of the prism?
27. What is the approximate volume of a rectangular room that has a floor that measures 15 feet by 11 feet and has ceilings that are 8 feet high?


Your neighbors Lena and Serena are looking for a refrigerator for their family and they have asked for your help. Above are the dimensions of 3 standard sizes of refrigerator. Some dimensions are given as a range. For example, the height of a small fridge can be anywhere between 60 inches to 75 inches. For those dimensions you can choose any number within the range to use in your calculation.
28. Approximately how many cubic inches of space are there in the small refrigerator?
29. Approximately how many cubic inches of space are there in the medium refrigerator?
30.Approximately how many cubic inches of space are there in the large refrigerator?


Lena and Serena heard that a refrigerator should hold about 4-6 cubic feet of groceries for every adult in the household. The volume you calculated above is in cubic inches.
31. If there are 12 inches in 1 foot, how many cubic inches are there in 1 cubic foot?
32. How many cubic feet of volume are there in the small, medium, and large refrigerators?
33. A local pet store sells 2 sizes of aquariums. Each has glass on the sides and the bottom, but not on the top.

Here are the dimensions of the 2 aquariums.


The smaller aquarium costs $\$ 24$.
What do you think would be a fair price for the larger aquarium?
34. Adjua is moving and estimates she has about 800 cubic feet of boxes. Calculate the volume of the moving truck in the diagram below to determine if it is large enough.


## Volume of Other Prisms

To calculate the volume of any prism, it works the same as it does when we calculate the volume of rectangular prisms. We calculate the area of the base and multiply that by the height of the prism.

Remember the height of a triangular prism is the distance between the 2 triangular ends. The height of the triangle face is the distance from the base of the triangle to the point opposite the base. That doesn't change no matter how the prism is standing.

For example, the height of this triangular prism is 24 ft . The height of the triangle face is 8 ft . The area of the triangle face is $1 / 2 \mathrm{bh} .1 / 2(8 \times 10)=40 \mathrm{sq} \mathrm{ft}$. If we multiply the area of the base by the height, we can calculate the volume. $40 \times 24=960$ cubic feet. .


Calculate the volume of the following prisms:
35. The diagram below shows the measurements for an attic space. What is the volume of the space?

36. What is the volume of this triangular prism?

37. Calculate the volume of these two shipping containers from the US Postal Service. (Note: You may round your answers to the nearest hundredth, which is two places after the decimal point)


The volume of shipping container $A$ is $\qquad$

The volume of shipping container $A$ is $\qquad$

## Volume of Cubes

## Building and Calculating Cube Roots

You may have heard of numbers being "cubed". For example: Five "cubed" equals 125.
The idea of numbers being cubed comes from volume. "Five cubed" means building a cube from a length of 5 units and calculating the volume of that cube. Since all the sides of the cube equal 5 , then 5 cubed equals 125.


Another way to write 5 cubed equals 125 is using an exponent. An exponent tells us how many times to multiply a number by itself. Exponents are written next to the number, but raised.

For example, $5^{3}$ means $5 \times 5 \times 5$.
3 is the exponent that tells us to multiply 5 times 5 times 5 .

$$
5^{3}=125
$$


38. Complete the table:

| Side length of cube | Volume of Cube |  |
| :---: | :---: | :---: |
| 1 | $1^{3}$ | 1 |
| 2 | $2^{3}$ | 8 |
| 3 | $3^{3}$ |  |
| 4 | $4^{3}$ | 125 |
| 5 | $5^{3}$ |  |
| 6 | $6^{3}$ |  |
| 7 | $7^{3}$ |  |
| 8 | $8^{3}$ |  |
| 9 | $9^{3}$ |  |
| 10 | $10^{3}$ |  |
| 11 | $11^{3}$ |  |
| 12 | $12^{3}$ |  |
| 2 |  |  |

The numbers in the right column of this table are called perfect cubes.
Perfect cubes are made when we cube whole numbers.
For example, 2,197 is a perfect cube because $13 \times 13 \times 13$ is 2,197 .

We can also go the other way. Starting with a cube, we can figure out the length of one side of that cube. The length of that edge is called the cube root.


The volume of this cube is 343 cubic units.


What is the length of one side of this cube?

7
$7 \times 7 \times 7=343$

We can write this as $7^{3}=343$, which can be read as " 7 cubed equals 343 ,"
OR
We can write $7=\sqrt[3]{343}$ which is read as "The cube root of 343 is 7 ."

There are several strategies for calculating the cube root of a number.

## Strategy \#1: Guess and Check

Let's say we want to calculate the cube root of 5832.
Another way to say this is:
Imagine a cube with an area of 5832 cubic units.
What is the length of one edge of the cube?
To calculate the cube root of a number we can ask, "What number can I multiply by itself 3 times to give me a cube with the volume I am looking for?"

Especially if you have a calculator, a strategy of guess-and-check can be quick.

Try a few different guesses to see how many tries it takes you to calculate the cube root of 5832.

- Record each attempt in the space below.
- After each attempt, ask yourself, "Should my next guess be smaller or larger than the number I just tried?"

| Guess <br> (Cube Root) | Number Squared |
| :---: | :---: |
| 13 | $13^{3}=2197$ |
|  |  |
|  |  |
|  |  |


| Guess <br> (Cube Root) | Number Squared |
| :--- | :--- |
|  |  |
|  |  |
|  |  |
|  |  |

## Strategy \#2: Use the cube root function on a calculator.



If you have a smartphone, you probably have a cube root button. Go into the calculator mode. Then turn your phone on its side and you will see more options. One of them will look like the symbol above.

Enter the number first and then press the button.

For example, to find the cube root of 343 , press 343 and then press
$\sqrt[3]{x}$

The TI-30XS is the calculator you are allowed to use for the HSE exam.

On the TI-30XS, there are a few additional steps. Here is how you find cube roots:

First press $\square$ This tells the calculator you want to find a cube root.

Then press 2nd This tells the calculator to use the green function of the next button you are going to press.

Then press


Then enter the number you want to find the cube root of. Then press enter.

```
To find the cube root of 343:
```

press 3 and then press

 then type 343, then press ENTER
$\qquad$
39. Use your calculator to calculate the cube root of the following numbers:

| Number | Cube Root |
| :---: | :---: |
|  | 729 |
| 8,000 |  |
|  | 12,167 |
| 24,389 |  |
|  | 0.512 |

Did the number you tried give you a cube root with many digits after the decimal point? If so, don't worry! It just means the number you entered was not a perfect cube.

Let's say we want to calculate the cube root of 90 .
There is no whole number that you can multiply by itself and then by itself again to get 90 . But there is a cube root of 90 .

You could use your calculator to determine a precise answer.
A cube root of 90 is 4.48140474656 , which means that each edge of this cube has a length of about 4.5 feet.

We can also use the perfect cubes to estimate non-perfect cubes.
The cube root of 125 is 5 and the cube root of 64 is 4 .
If we are looking for a cube with a volume of 90 cubic units, a side of 4 is too small and a side of 5 is too big. So we can estimate that the cube root of 90 is between 4 and 5 .

To help us visualize that, let's imagine that we have 90 blocks that we want to arrange into a cube with a volume of 90 .


The largest cube we can create is a $4 \times 4 \times 4$ cube.

We have a cube with a volume of 64 and 26 extra blocks.

The next largest cube would be
$5 \times 5 \times 5$ ( 125 blocks). We would need 61 more blocks for that. 26 blocks are not enough.

That means the cube root of 90 is between 4 and 5 .
40. Determine the cube root of each of the following numbers:
a. The cube root of 5,832 is $\qquad$ .
b. If the volume of a cube is 9,261 , the length of one side of the cube is $\qquad$ .
c. What number to the third power is 27,000 ? $\qquad$
d. $\qquad$ $\times$ $\qquad$ $\times$ $\qquad$ $=15.625$.
41. The cube root of 636 is between:
A. 6 and 7
B. 7 and 8
C. 8 and 9
D. 9 and 10
42. A cube has a volume of 7,200 cubic inches. What is the approximate length of each of its sides?
A. between 16 inches and 17 inches
B. between 17 inches and 18 inches
C. between 18 inches and 19 inches
D. between 19 inches and 20 inches

## Volume of Square Pyramids

We can use what we know about calculating the volume of cubes to calculate the volume of square pyramids.

The cube and the pyramid below are similar.


The cube and the pyramid are the same height.


If we stack one on top of the other, we can see that the square base of the cube is the same size as the square base of the pyramid.


The pyramid is filled with clear syrup.
How many full pyramids do you think it will take to fill the cube?

Explain your guess below.
$\qquad$
$\qquad$
$\qquad$


We fill the pyramid a third time, this time with yellow cooking oil.

Complete the following sentences to describe the relationship between the volume of the pyramid and the volume of the cube.

- The volume of the cube is $\qquad$
the volume of the pyramid.
- The volume of the pyramid is $\qquad$

the volume of the cube.

When we compare the volume of the cube and the volume of the pyramid with the same height and the same square base, we can observe a few things.

The volume of the cube is larger. How much larger?
The volume of a cube is 3 times more than the volume of a pyramid.
The volume of the pyramid is smaller. How much smaller?
The volume of a pyramid is $1 / 3$ (one-third) the volume of a cube
To calculate the volume of any square pyramid, we can:

- Imagine a rectangular prism of the same height and base
- Calculate the volume of that rectangular prism
- Divide the volume of the cube by 3

You learned that you can calculate the volume of any rectangular prism - including a cube using the formula $\boldsymbol{V}=\mathbf{l w h}$ where $V$ is the volume, $l$ represents the length, $w$ represents the width, and $h$ represents the height.


The formula sheet you will get when you take your high school equivalency exam might look something like this:
$V=1 / 3 B h \quad \begin{gathered}\text { where } B \text { represents the area of the base of the } \\ \text { pyramid and } h \text { represents the height of the }\end{gathered}$ pyramid

This formula may look different, but if we break it down, it is the same.

To calculate the area of the base of the pyramid we multiply the length by the width. So l x w is the same as B.

Multiplying a number by $1 / 3$ is equivalent to dividing it by 3 .


It might seem like a strange idea to say that multiplying a number by $1 / 3$ is equivalent to dividing it by 3 . If you're not sure it's true, try for yourself!
43. $27 \div 3$
$48.1 / 3 \times 145$
$44.1 / 3 \times 27$
49. $245 \div 3$
$45.369 \div 3$
$50.1 / 3 \times 245$
46. $1 / 3 \times 369$
51. $1515 \div 3$
47. $145 \div 3$
52. $1 / 3 \times 1515$

Because both methods will give you the same answer, you should decide for yourself which way of using the formula makes more sense to you and use that one.

$$
\mathrm{V}=(\mathrm{lw}) \mathrm{h}
$$

$$
V=1 / 3(B) h
$$

53. The Great Pyramid at Giza is 147 meters high and has a base length of 231 meters. What is the approximate volume of the pyramid?


## Volume of Prisms and Pyramids - Answer Key

1. 10 cubic centimeters, $10 \mathrm{cu} . \mathrm{cm}$., $10 \mathrm{~cm}^{3}$

7 cubic inches, 7 cu. in., 7 in $^{3}$
12 cubic centimeters, $12 \mathrm{cu} . \mathrm{cm}$., $12 \mathrm{~cm}^{3}$
8.5 cubic inches, 8.5 cu . in., $8.5 \mathrm{in}^{3}$
$1 / 2$ cubic meters, $1 / 2$ cu. m., $1 / 2 \mathrm{~m}^{3}$
36 cubic feet, $36 \mathrm{cu} . \mathrm{ft}^{2}, 36 \mathrm{ft}^{3}$
12 cubic miles , 12 cu. mi., $12 \mathrm{mi}^{3}$
2. Choices $C$ and D. You can use in ${ }^{3}$ or $c u$. in. to abbreviate cubic inches. The exponent ${ }^{3}$ refers to a cube's three dimensions: Length, height and width.
3. Choices C, D, and E. You can use $\mathrm{cm}^{3}, \mathrm{cu} . \mathrm{cm}$., or cc to abbreviate cubic centimeters. The exponent ${ }^{3}$ refers to a cube's three dimensions: Length, height and width.
4. There are many possible answers. As one example, a marshmallow has a volume of about 30 cubic centimeters.
5. There are many possible answers. As one example, a 20 oz cup has a volume of about 600 cubic centimeters.
6. There are many possible answers. As one example, a can of soda has a volume of about 20 cubic inches.
7. There are many possible answers. As one example, an average bathtub has a volume of about 10,000 cubic inches.
8. There are many possible answers. As one example, an average bathtub has a volume of about 6 cubic feet.
9. There are many possible answers. As one example, a queen-sized mattress has a volume of about 30 cubic feet.
10. There are many possible answers. As one example, a room that is 15 feet long, 10 feet wide, and 8 feet tall has a volume of 1200 cubic feet.
11. There are many possible answers. As one example, the cargo bed of a fully loaded, full size standard pickup truck has a volume of approximately $21 / 2$ cubic yards.
12. There are many possible answers. As one example, an Olympic-sized swimming pool has a volume of approximately 3,300 cubic yards.

# Three-Dimensional Geometry (Part 1) 

13. 12 inches
14. 6 inches
15. 72 cubes
16. The space inside the box can hold 432 cubes. 72 cubes can fit in the space along the bottom of the box. Since the box is 6 cubes high, 6 layers of 72 can fit in the box. 72 x $6=432$. You can also think about the side of the box. Since 6 cubes fit along the width and 6 cubes fit along the height, it would take 36 cubes to fill the space along the side of the box. Since the length is 12 , that means 12 layers of 36 cubes could fit in the space inside the box. $12 \times 36=432$.
17. 5 possible answers include: $1 \times 3 \times 8=24,2 \times 4 \times 7=56,3 \times 4 \times 6=72,2 \times 6 \times 7=84,3 \times 4 \times 8=96$
18. 3 possible answers include: $4 \times 6 \times 8=192.3 \times 8 \times 9=216.6 \times 8 \times 9=432.4 \times 6 \times 8=192$
19. Here are some possible answers: 1 in by 6 inches by 8 inches. 2 inches by 3 inches by 8 inches. 3 inches by 4 inches by 4 inches. 4 inches by 6 inches by 2 inches.
20. The surface area will depend on your answer to the previous question. The surface area of a 1 by 6 by 8 rectangular prism is 124 sq in . The surface area of a 2 by 3 by 8 rectangular prism is 92 sq in . The surface area of a 2 by 4 by 6 rectangular prism is 88 sq in. The surface area of a 3 by 4 by 4 rectangular prism is 80 sq in.
21. A box in the shape of a cube that measures 4 inches on each side would have a volume of 64 cubic inches and a surface area of 96 square inches.
22. $24+24+72+60=180$ cubic feet
23. There are 27 cubic feet in 1 cubic yard.
24.7 cubic yards.
24. 5,028 ft ${ }^{3}$
25. 12 ft . If a rectangular prism has a length of 10 feet, a width of 12 feet and a height of 12 feet, it will have a volume of 1,440 cubic feet.
26. 1,320 cubic feet
27. It depends on the numbers you choose, but the smaller fridge will have somewhere between 41,760 cubic inches and 59,400 cubic inches.
28. It depends on the numbers you choose, but the medium fridge will have somewhere between 64,350 cubic inches and 89,100 cubic inches.
29. It depends on the numbers you choose, but the large fridge will have somewhere between 64,800 cubic inches and 90,720 cubic inches.
30. There are 1,728 cubic inches in 1 cubic foot.
31. This answer depends on the number you choose, but to calculate the answer, divide the number of cubic inches inside each fridge by 1,728 . That will tell you the number of cubic feet inside each fridge.
32. IF YOU ARE LOOKING FOR A HINT: If you are not sure how to begin, you might start by finding the surface area and the volume for each aquarium.

The surface area of the larger aquarium is $4 x$ larger than the surface area of the smaller aquarium. If the smaller aquarium costs $\$ 24$, then one way to think about the cost of the larger one is to multiply $\$ 24 \times 4$, which equals $\$ 96$. The volume of the larger aquarium is $8 x$ larger than the volume of the smaller one, so another argument can be made that the larger one should cost $\$ 192$ ( $\$ 24 \times 8$ ). $\$ 96$ and $\$ 192$ are 2 possible answers. There are other reasonable answers.
34. The truck has a volume of 840.07 cubic feet. It should be ok to move Adjua's boxes.
35. 272 cubic yards
36. 1,196 cubic meters
37. The volume of package $A$ is $563.06 \mathrm{in}^{3}$. The volume of package $B$ is $402.61 \mathrm{in}^{3}$.
38.

| $1^{3}$ | 1 | $7^{3}$ | 343 |
| :---: | :---: | :---: | :---: |
| $2^{3}$ | 8 | $8^{3}$ | 512 |
| $3^{3}$ | 27 | $9^{3}$ | 729 |
| $4^{3}$ | 64 | $10^{3}$ | 1000 |
| $5^{3}$ | 125 | $11^{3}$ | 1331 |
| $6^{3}$ | 216 | $12^{3}$ | 1728 |

39. 

| Number | Cube Root | Number | Cube Root |  |
| :---: | :---: | :---: | :---: | :---: |
| 729 | 9 | 24,389 | 29 |  |
| 8,000 | 20 | .512 | 0.8 |  |
| 12,167 | 23 |  |  |  |

40. 

a. 18
b. 21
c. 30
d. 2.5
41. Choice C
42. Choice D
43.9
44.9
45.123
46.123
47. 48.33
48.48.33
49. 81.66
50.81 .66
51. 505
52. 505
53. $2,614,689$ cubic meters. To calculate the area of the square base we multiply $231 x$ 231, which tells us the area of the base is 53,361 square meters, We multiply the area of the base times the height, so $53,361 \times 147$, which equals $7,844,067$. We divide 7 , 844,067 by 3 and get 2,614,689 cubic meters.

## Volume of Round Objects

## Volume of Cylinders

Here are two identical sheets of paper. The shorter side of each paper is 9 inches and the longer side is 12 inches.


Imagine we roll each sheet into a cylinder.
One sheet is rolled the long way. One sheet is rolled the short way.


Which cylinder do you think has a greater volume?

Check one:


I think the taller cylinder will hold more
$\square$ I think the wider cylinder will hold more .I think the two cylinders will hold the same amount.

Explain your reasoning.

$\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$

To answer our question, we filled the taller cylinder all the way to the top with dried beans. Then we put the wider cylinder around the taller one. We'll pull out the taller cylinder and see how full the wider one is. (Try it for yourself!)



When we empty the taller cylinder filled to the top with beans into the wider cylinder, we still have a lot more room to add more beans. So the wider cylinder has a larger volume!


Are you surprised by the result? Let's learn more about how to calculate the volume of a cylinder to understand why the wider cylinder has a greater volume than the taller one.

A cylinder is not a prism because it has a curved side, but it does have a lot in common with a prism:

- A cylinder has a flat top and a flat base.
- The base is the same as the top.
- From the base to the top, the shape stays the same.

Because of these similarities, our method for calculating the volume of a cylinder is similar to our method for calculating the volume of a prism.


To find the volume of a cylinder, we find the area of the circular base and multiply it by the height of the cylinder.


The volume of a stack of quarters is the area of the face of 1 quarter multiplied by the height of the stack.

> The formula for the area of a circle is $A=\pi r^{2}$ so the formula for the volume of a cylinder is $V=\pi r^{2} \times h$.

The HSE Math Formula Sheet writes the formula as $V=\pi r^{2} h$.

Calculate the volume of the following cylinders.
1.

2.

3. Which of the following cylinders has the largest volume?
A. $\quad \begin{aligned} & h=24 \\ & r=1 \mathrm{~cm}\end{aligned}$

B.


$$
\begin{aligned}
& \mathrm{h}=12 \mathrm{~cm} \\
& \mathrm{r}=2 \mathrm{~cm}
\end{aligned}
$$

C.


$$
\begin{aligned}
& \mathrm{h}=8 \mathrm{~cm} \\
& \mathrm{r}=3 \mathrm{~cm}
\end{aligned}
$$

D.


$$
\begin{aligned}
& \mathrm{h}=6 \mathrm{~cm} \\
& \mathrm{r}=4 \mathrm{~cm}
\end{aligned}
$$

4. In 2010, a cylindrical sinkhole appeared in Guatemala City, Guatemala. The hole was 90 meters deep and 20 meters in diameter. What is the volume of the cylindrical hole?

5. An oatmeal company sells its oatmeal in a cylindrical container that has a diameter of 4 inches and is 7 inches tall. They want to change their package and sell oatmeal in rectangular prism boxes. They want the height of the box to be the same as the height of the cylinder. They also want the volume of the box to be as close to the volume of the cylindrical package as possible. Which of the following would be the best dimensions for the new packaging?
A. 4 inches, 4 inches, 7 inches
B. 5 inches, 3 inches, 7 inches
C. 6 inches, 2 inches, 7 inches
D. 5 inches, 2.5 inches, 7 inches

Explain your choice.

[^6]Remember the two paper cylinders we looked at earlier?
Let's look at them now that you have learned more about finding the volume of a cylinder.


The circumference of a circle is approximately 3 times the length of its diameter.
If the circumference of a circle is 9 inches, then its diameter will be about 3 inches.
If the circumference of a circle is 12 inches, then its diameter will be about 4 inches.
Using those estimations, calculate the volume of each paper cylinder.
6. I think the volume of the taller cylinder is about...
7. I think the volume of the wider cylinder is about...

Which measurement has a bigger impact on the volume of a cylinder, the radius or the height?
8. Imagine a cylinder with a radius of 4 in and a height of 10 in .

What would happen to the volume of the cylinder if we doubled the height?
What would happen to the volume of the cylinder if we doubled the radius?
Fill in the table below to find out.

| CYLINDER | RADIUS (in) | HEIGHT (in) | VOLUME |
| :---: | :---: | :---: | :---: |
| Cylinder | 4 | 10 |  |
| Double the height | 4 | 20 |  |
| Double the radius | 8 | 10 |  |

What do you notice?
9.

| CYLINDER | RADIUS (in) | HEIGHT (in) | VOLUME |
| :---: | :---: | :---: | :---: |
| Cylinder | 5 | 7 |  |
| Double the height | 5 | 14 |  |
| Double the radius | 10 | 7 |  |

10. 

| CYLINDER | RADIUS (in) | HEIGHT (in) | VOLUME |
| :---: | :---: | :---: | :---: |
| Cylinder | 3 | 4 |  |
| Triple the height | 3 | 12 |  |
| Triple the radius | 9 | 4 |  |

## Volume of Cones

Both cylinders and cones have a circular base. We can use what we know about finding the volume of cylinders to find the volume of cones.

The cone and cylinder below are similar.


Side View:
The cone and the cylinder are the same height.


## Top View:

The cone and the cylinder have the same size circle as a base. That means that their bases have the same diameter and the same radius.


The cone is filled with water. (The water is dyed blue to help you see it.)

How many full cones do you think it will take to fill the cylinder? Explain your guess below.
$\qquad$
$\qquad$
$\qquad$


The cone is completely filled a second time, this time with clear corn syrup so we can see the difference.

We fill the cone a third time, this time with cooking oil.


The volume of this cylinder is divided into 3 equal layers.

Each layer is equal to the volume of the cone.

Complete the following sentences to describe the relationship between the volume of the cone and the volume of the cylinder.

- The volume of the cylinder is $\qquad$ the volume of the cone.
- The volume of the cone is $\qquad$ the volume of the cylinder.

The volume of a cylinder is 3 times more than the volume of a cone with the same height and radius.

The volume of a cone is $1 / 3$ the volume of a cylinder with the same height and radius.
To calculate the volume of a cone, we can find the volume of a cylinder and divide it by 3 ! In the last section, you learned that you can use the formula $V=\pi r^{2} h$ to calculate the volume of a cylinder, where $V$ is the volume, $r$ is the radius and $h$ is the height.

$$
\begin{aligned}
& \text { To find the volume of a cone... } \\
& V=\frac{\left(\pi r^{2} h\right)}{3} \text { we use the formula for } \\
& \text { the volume of a cylinder }
\end{aligned}
$$

The HSE Formula sheet has the same formula, but writes it as:

$$
V=1 / 3 \pi r^{2} h
$$

which means the volume of the cone is equal to one-third the volume of the cylinder.
Both methods are equivalent and will give you the same answer.
11. An ice cream cone has a height of 15.5 cm and a diameter of 6 cm . What is the volume of the cone?
A. 146.01 cubic cm
B. 438.03 cubic cm
C. 584.04 cubic cm
D. $1,752.12$ cubic cm
12. A large cone-shaped candle has a height of 12 inches and a radius of 3 inches. What is the volume of the candle?

## Volume of a Sphere

Figuring out the volume of a sphere is difficult because it doesn't have any flat sides. One way mathematicians have thought about it is to imagine that it has lots of small flat sides. It turns out that thinking about it in this way actually leads to the correct formula. Here is a picture of what a sphere would look like if it had lots of small flat sides instead of a smooth curved surface.

Imagine cutting a wedge out of the sphere and opening up the wedge so that each piece of the wedge is a pyramid.
The height of each pyramid is the radius of the sphere.


We can use what we know about calculating the volume of a pyramid and the surface area of a sphere to determine the volume of a sphere.

We know from our previous work that the formula for calculating the volume of a pyramid is

$$
\mathbf{V}=\frac{B h}{3} \text {, where } B \text { equals the surface area of the base. }
$$

Each pyramid has a height that is equal to the radius of the sphere, so we can replace the height with the radius in the formula and say

$$
\mathbf{V}=\frac{B r}{3}
$$

Rectangles cover the surface of the sphere. Those rectangles are the bases of the pyramids we created. Since those rectangles cover the entire surface area of the sphere, the total area of base of every pyramid is equal to the surface area of the entire sphere.

The formula for calculating the surface area of a sphere is

$$
S A=4 \pi r^{2}{ }_{\text {so }} B=4 \pi r^{2}
$$

In our volume formula, if we replace $B$ with $4 \pi r^{2}$, our formula can be written as

$$
\mathbf{V}=\frac{4 \pi r^{2} \times r}{3}
$$

$r^{2}$ means $r \times r$ so $r^{2} \times r$ means $r \times r \times r$ which we can rewrite as $r^{3}$

$$
\mathbf{V}=\frac{4 \pi r^{3}}{3}
$$

which is the formula for calculating the volume of a sphere!
13. Use the given radius and complete the chart to calculate the area of each sphere.

| $r$ | $r^{3}$ | $\pi r^{3}$ | $4 \pi r^{3}$ | $\frac{4 \pi r^{3}}{3}$ | Volume of <br> Sphere |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 4 |  |  |  |  |  |
| 5 |  |  |  |  |  |
| 6 |  |  |  |  |  |

14. In an early scene in the movie Raiders of the Lost Ark, the main character Indiana Jones has to avoid a giant boulder rolling after him.

If the boulder has a diameter of 7 meters, what is the approximate volume of the boulder?

15.


This sphere has a diameter of 36 cm . What is the volume of the sphere?
16.


This bowl is a hemisphere and has a radius of 6 inches.
What is the volume of the bowl?
17. This cup was designed with a hemisphere base and a cylindrical top and has a diameter of 20 cm . What is the volume of the cup?

A. $5,861 \mathrm{cu} . \mathrm{cm}$
B. $7,955 \mathrm{cu} . \mathrm{cm}$
C. $31,818 \mathrm{cu} . \mathrm{cm}$
D. $48,565 \mathrm{cu} . \mathrm{cm}$
18. The diagram below contains a cylinder, sphere, and cone.


Which of the following statements are true?
A. The volume of the cylinder is less than the volume of the sphere plus the volume of the cone.
B. The volume of the cylinder is equal to the volume of the sphere plus the volume of the cone.
C. The volume of the cylinder is more than the volume of the sphere plus the volume of the cone.

## Story Tables

So far in this section you have been working on calculating the volume of a sphere when you are given either the radius (or the diameter) of the sphere. But we can also figure out the radius of a sphere when we are given the volume.

There are several strategies we can use. One of them is called a story table. Story tables are powerful tools for solving algebraic equations. You've already used them a few times as you have been working through this packet. The way it works is to take the formula and turn it into a series of steps.
The formula for calculating the volume of a sphere is $\frac{4 \pi r^{3}}{3}$.
To make a story table for this formula, we need to tell the story of $r$. What are the steps we take when we are using the formula?

- First, we cuber.
- Next, we multiply by $\pi$.
- Next, we multiply by 4.
- Finally, we divide by 3 and get the volume of the sphere.

To make a story table, we make a column for each step, starting with $r$. In the last column, we write the volume we are targeting.

If we hit our target, excellent! If we don't hit our target, we write NO in the last column and try a different value for $r$. After each guess, it is important to ask "Should my next guess be smaller or larger than the number I just tried?"

For example, let's say we are looking for the radius of a sphere that has a volume of 7,234.56 cubic meters. We would create a story table like the one below. And then we make a guess for $r$. What could $r$ be?

Let's try 8 . We try a number by putting it in the $r$ column and then performing all the steps.
19. Since 8 didn't work, try another number.

Should it be larger or smaller than 8 ?
Keep going until you find a value for $r$ that hits the target.

| $r$ | $r^{3}$ | $\pi r^{3}$ | $4 \pi r^{3}$ | $\frac{4 \pi r^{3}}{3}$ | $\mathbf{7 , 2 3 4 . 5 6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 512 | $1,607.68$ | $6,430.72$ | $2,143.57$ | NO |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

20. 

The Unisphere in Flushing, Queens is a spherical stainless steel representation of the Earth. It was built for the 1964 New York World's Fair.

The Unisphere has a volume of 904,320 cubic feet.

What is the radius of the Unisphere?


| $r$ | $r^{3}$ | $\pi r^{3}$ | $4 \pi r^{3}$ | $\frac{4 \pi r^{3}}{3}$ | 904,320 |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

21. 



The Epcot Center geodesic sphere in Florida has a volume of $91,905.71$ cubic yards.

What is the radius of the Epcot Center sphere?

|  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

We can also use story tables with cylinders.
The formula for calculating the volume of a cylinder is $V=\pi r^{2} \times h$
This formula uses 2 measurements: the radius of the cylinder and the height of the cylinder.
If we are given the volume of the cylinder and one of those measurements, we can use a story table to find the missing measurement.

For example, the diagram below shows an above ground swimming pool with a height of 4 ft . If the volume of the pool is $1,808.64$ cubic feet, what is the radius of the pool?


Since this formula has 2 measurements, we can add a column to help us keep track of the information we have been given. In this case, we know the height of the cylindrical pool.

Now we can proceed the same way we did before. What could the radius be? Could it be 20?
22. Keep trying until you find a radius that hits our target. Remember to ask yourself, "Should my next guess be smaller or larger than the number I just tried?"

| Height | $\boldsymbol{r}$ | $\boldsymbol{r}^{2}$ | $\pi \boldsymbol{r}^{2}$ | $\pi \boldsymbol{r}^{2} \times h$ | $1,808.64$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{4 ~ f t}$ | 20 | 400 | 1,256 | 5,024 | NO |
| $\mathbf{4 ~ f t}$ |  |  |  |  |  |
| $\mathbf{4 ~ f t}$ |  |  |  |  |  |
| $\mathbf{4 ~ f t}$ |  |  |  |  |  |
| $\mathbf{4 ~ f t}$ |  |  |  |  |  |
| $\mathbf{4 ~ f t}$ |  |  |  |  |  |
| $\mathbf{4 ~ f t}$ |  |  |  |  |  |

What if we were given the radius and the volume and needed to find the height of a cylinder?

Let's say we have a cylinder with a radius of 3 inches and a volume of
 1,017.36 cubic inches.

Could the height be 11 inches?
23. Keep trying until you find a height that hits our target.

| $\boldsymbol{h}$ | $\boldsymbol{r}$ | $\boldsymbol{r}^{2}$ | $\boldsymbol{\pi r ^ { 2 }}$ | $\boldsymbol{\pi r ^ { 2 } \times h}$ | $\mathbf{1 , 0 1 7 . 3 6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 11 | 3 in | 9 | 28.26 | 339.12 | NO |
|  | 3 in | 9 | 28.26 |  |  |
|  | 3 in |  |  |  |  |
|  | 3 in |  |  |  |  |
|  | 3 in |  |  |  |  |
|  | 3 in |  |  |  |  |
|  | 3 in |  |  |  |  |

## Volume of Round Objects: Cylinders, Cones, and Spheres - Answer Key

1. 2411.52 cubic centimeters
2. 602.88 cubic centimeters
3. Choice D.

- Cylinder A has a volume of 75.36 cubic cm
- Cylinder B has a volume of 150.72 cubic cm
- Cylinder C has a volume of 226.08 cubic cm
- Cylinder D has a volume of 301.44 cubic cm

What do you notice about the volumes of these 4 cylinders?
4. 28,260 cubic meters
5. Choice D. The cylindrical package has a volume of 87.92 cubic inches. A rectangular prism that is 5 inches by 2.5 inches by 7 inches has a volume of 87.5 cubic inches.
6. The volume of the taller cylinder is about 84.78 cubic inches.
7. The volume of the wider cylinder is about 113.04 cubic inches.
8.

| CYLINDER | RADIUS | HEIGHT | VOLUME |
| :---: | :---: | :---: | :---: |
| Cylinder | 4 | 10 | $502.4 \mathrm{cu} . \mathrm{in}$. |
| Double the height | 4 | 20 | $1,004.8 \mathrm{cu} . \mathrm{in}$. |
| Double the radius | 8 | 10 | $2,009.6 \mathrm{cu} . \mathrm{in}$. |

$1,004.8$ is $2 x$ as big as 502.4
$2,009.6$ is $4 x$ as big as 502.4
9.

| CYLINDER | RADIUS (in) | HEIGHT (in) | VOLUME |
| :---: | :---: | :---: | :---: |
| Cylinder | 5 | 7 | $549.5 \mathrm{cu} . \mathrm{in}$. |
| Double the height | 5 | 14 | $1,099 \mathrm{cu} . \mathrm{in}$. |
| Double the radius | 10 | 7 | $2,198 \mathrm{cu} . \mathrm{in}$. |

1,099 is $2 x$ as big as 549.5
2,198 is 4 x as big as 549.5
10.

| CYLINDER | RADIUS (in) | HEIGHT (in) | VOLUME |  |
| :---: | :---: | :---: | :---: | :---: |
| Cylinder | 3 | 4 | $113.04 \mathrm{cu} . \mathrm{in}$. |  |
| Triple the height | 3 | 12 | $339.12 \mathrm{cu} . \mathrm{in}$. |  |
| Triple the radius | 9 | 4 | $1,017.36 \mathrm{cu} . \mathrm{in}$. |  |
| 339.12 is $3 x$ as big as 113.04 <br> $1,017.36$ is $9 x$ as big as 113.04 |  |  |  |  |

11. Choice A.
12. 113.04 cubic inches
13. 

| $r$ | $r^{3}$ | $\pi r^{3}$ | $4 \pi r^{3}$ | $\frac{4 \pi r^{3}}{3}$ | Volume |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 64 | 200.96 | 803.84 | 267.95 | 267.95 |
| 5 | 125 | 392.5 | 1,570 | 523.33 | 523.33 |
| 6 | 216 | 678.24 | $2,712.96$ | 904.32 | 904.32 |

14. The boulder would have a volume of 179 cubic meters.
15. 24416.64 cubic cm
16. 452.16 cubic inches
17. Choice $A$. The volume of the cylindrical part of the cup is 3,768 cubic cm . The volume of the hemisphere part of the cup is 2093 cubic $\mathrm{cm} .3,768$ cubic $\mathrm{cm}+2,093 \mathrm{cubic} \mathrm{cm}=$ 5,861 cubic cm.
18. Choice $B$. The volume of the cylinder is equal to the volume of the sphere and the volume of the cone combined.
19. 12 m . A sphere with a volume of $7,234.56$ cubic meters has a radius of 12 m .
20.60 feet. A sphere with a volume of 904,320 cubic meters has a radius of 60 ft .
20. 28 yards. A sphere with a volume of $91,905.70$ cubic yards has a radius of 28 yds .
21. 12 ft . A cylinder with a volume of $1,808.64$ cubic feet and a height of 4 ft has a radius of 12 ft .
22. 36 inches. A cylinder with a volume of $1,017.36$ cubic inches and a radius of 3 in has a height of 36 inches

## Is It Surface Area or Is It Volume?

In this packet, you have done a lot of work to understand different methods for measuring the surface area and volume of different types of three-dimensional objects.

As a review of when you might use the different methods, imagine the situations below and decide whether each is a measure of surface area (SA) or volume (V).

1) Painting the outside of a house.
2) Determining how much fabric you need to construct a tent.
3) Filling a closet with boxes of tissue paper.
4) How much space there is underneath a table.
5) Filling a can with paint.
6) The amount of cardboard it takes to make a box.
7) How much sand is needed to fill a sandbox.
8) How many boxes fit inside a moving van
9) How much material you need to make a rectangular milk container.
10) How many mirrors are needed to cover a disco ball.
11) How much candy will fit inside a piñata.
12) How much frosting to put on a cake.
13) How much water fills up inside a laundry machine.

## Is It Surface Area or Is It Volume? - Answer Key

1) Surface area
2) Surface area
3) Volume
4) Volume
5) Volume
6) Surface area
7) Volume
8) Volume
9) Surface area
10) Surface area
11) Volume
12) Surface area
13) Volume

## Formulas for Three-Dimensional Geometry

| VOLUME |  |  |
| :---: | :---: | :---: |
| Rectangular Prisms | $\mathrm{V}=1 w h$ |  |
| Right Prisms (like triangular prisms) | $\mathrm{V}=\boldsymbol{B} \boldsymbol{h}$ |  |
| Pyramid | $\mathrm{V}=\frac{l w h}{3}$ | $V=1 / 3 B h$ |
| Cylinder | $V=\pi r 2 \times h$ |  |
| Cone | $\mathrm{V}=\frac{\pi r^{2} h}{3}$ | $V=1 / 3 \pi r^{2} h$ |
| Sphere | $\mathbf{V}=\frac{4 \pi r^{3}}{3}$ | $\mathrm{V}=\frac{4}{3} \pi r^{3}$ |

## SURFACE AREA

| Rectangular Prisms | $S A=2 l w+2 l h+2 w h$ | $S A=2 B+p h$ |
| :--- | :---: | :---: |
| Right Prisms | $\mathrm{SA}=2 B+p h$ |  |
| Pyramid | $\mathrm{SA}=B+(1 / 2 b h \times 4)$ | $\mathrm{SA}=B+1 / 2 p s$ |
| Cylinder | $\mathrm{SA}=2 \pi r h+2 \pi r^{2}$ |  |
| Cone | $\mathrm{SA}=\pi r s+\pi r^{2}$ |  |
| Sphere | $\mathrm{SA}=4 \pi r^{2}$ |  |

$\boldsymbol{r}=$ radius, $\boldsymbol{h}=$ height, $\boldsymbol{p}=$ perimeter, $\boldsymbol{B}=$ area of the base, $\boldsymbol{\pi} \cong 3.14, \boldsymbol{s}=$ slant height

## Test Practice Questions

Answer the following questions. You can check your answers in Test Practice Questions Answer Key. These questions will review what you have learned in this packet.

1) The volume of a cube is 125 cubic inches. What is the total surface area?
A. 25 square inches
B. 100 square inches
C. 150 square inches
D. 250 square inches
2) A baseball is a sphere with a diameter of approximately 2.9 inches. What is the approximate surface area of the baseball?

A. 6.6 square inches
B. 26.4 square inches
C. 105.6 square inches
D. 120.5 square inches
3) Which of these rectangular prisms has the largest surface area?
A. A rectangular prism that measures 8 cm by 12 cm by 3 cm .
B. A rectangular prism that measures 10 cm by 8 cm by 5 cm .
C. A rectangular prism that measures 8 cm by 6 cm by 7 cm .
D. A rectangular prism that measures 22 cm by 3 cm by 2 cm .
4) Which of these solids has the greatest volume?
A.

C.

B.

D.

5) Which of these cylinders has the largest volume?
A. A cylinder with a height of 12 ft and a radius of 3 ft
B. A cylinder with a height of 3 ft and a radius of 12 ft
C. A cylinder with a height of 6 ft and a radius of 10 ft
D. A cylinder with a height of 10 ft and a radius of 6 ft
6) This spherical beach ball has a surface area of 1808.64 square inches.

What is the volume of the beach ball?
A. 602.9 cubic inches
B. 904.3 cubic inches
C. 4,069.4 cubic inches
D. $7,234.6$ cubic inches

7) A box of cereal is a rectangular prism and has a length of 7.5 inches, a width of 2.5 inches, and a height of 12 inches. A different box of cereal has a length of 6 inches and a width of 3 inches. If both cereal boxes have the same volume, what is the height of the second cereal box?
A. 12.5 inches
B. 18 inches
C. 37.5 inches
D. 225 inches
8) Adela wants to paint a cylindrical tank that they bought to store rainwater in their garden. The tank is open at the top and they do not need to paint the bottom of the tank. They only need to paint the outside of the tank.


If one gallon can of paint covers 400 square feet, how many cans will they need to buy for the job?
A. 4
B. 5
C. 6
D. 7
9) Shonei made a box by cutting out 4-inch squares from each corner of a rectangular piece of cardboard and then folding the sides up.


The volume of the box is 360 cubic inches. The length of the shorter side of the box is 6 inches, what is the length of the longer side?
A. 3.75 inches
B. 7.5 inches
C. 15 inches
D. 60 inches
10) The parks department needs to install fencing to protect some new trees. When the fencing is in place around each tree, the diameter of the fencing is 1 foot. If each fence needs to be 3 feet tall, approximately how many square feet of fencing are needed to protect 5 trees?
A. 9.4 square feet
B. 47.1 square feet
C. 48.7 square feet
D. 94.2 square feet

11) The total volume of the two rectangular prisms in the diagram below is 900 cubic centimeters. What is the volume of the light colored rectangular prism?

A. 660 cubic cm
B. 820 cubic cm
C. 840 cubic cm
D. 1560 cubic cm

## 12) Part One

The Quetzal chocolate company sells chocolate bars in triangular prism boxes. The triangular faces are made up of equilateral triangles that measure 60 mm on each side. What is the approximate surface area of the box?

A. $12,480 \mathrm{sq} \mathrm{mm}$
B. $21,120 \mathrm{sq} \mathrm{mm}$
C. $57,120 \mathrm{sq} \mathrm{mm}$
D. $63,360 \mathrm{sq} \mathrm{mm}$

## Part Two

The Quetzal chocolate company sells the boxes shown above in packs of 9 boxes. What is the total volume of the 9 boxes of chocolate in this pack?
E. 468,000 cubic mm
F. 2,190,200 cubic mm
G. 4,212,000 cubic mm
H. $8,424,000$ cubic mm
(Note: Diagram not drawn to scale)

13) Which of the following dimensions for a rectangular prism would result in a volume of 96 cubic inches?
A. Width: 8 inches, Length: 1 inch, Height: 12 inches
B. Width: 19 inches, Length: 5 inches, Height: 1 inches
C. Width: 21 inches, Length: 4 inches, Height: 12 inches
D. Width: 32 inches, Length: 32 inch, Height: 32 inches

## 14) Part One

Shirley is making a candle in the shape of a pyramid as shown in the diagram below. If the candle has a height of 15 cm , what is the volume of the candle?

A. 240 cubic centimeters
B. 1,280 cubic centimeters
C. 1,365 cubic centimeters
D. 3,840 cubic centimeters

## Part Two

Shirley wants to wrap the candle in paper. If the pyramid has a slant height of 17 cm , how many square centimeters would it take to cover the surface of the pyramid, including the base?
A. 267 square centimeters
B. 544 square centimeters
C. 800 square centimeters
D. 1,344 square centimeters

15) The container below has a diameter of 14 feet and a combined height of 32 feet. What is the total volume of the container?

16) An office has a water cooler with a cylindrical water container. The diameter of the container is 10 inches and the height of the container is 18 inches.

B. 150
C. 200
D. 300

Note: Diagrams not drawn to scale
17) The two diagrams below represent two gardening beds in Alex's garden.


Note: Diagrams not drawn to scale
The larger garden bed is completely filled with soil. Alex takes soil from the larger garden bed to completely fill the smaller garden bed. How many cubic feet of soil will there be left in the larger garden bed?
A. 72 cubic feet
B. 288 cubic feet
C. 360 cubic feet
D. 432 cubic feet
18) A spherical disco ball has a diameter of 14 inches.

What is the surface area of the sphere?
A. $615.44 \mathrm{sq} . \mathrm{in}$.
B. $1,230.88 \mathrm{sq}$. in.
C. $1,436.03$ sq. in.
D. $2,461.76 \mathrm{sq} . \mathrm{in}$.
19) The diagram below shows three tennis balls in a cylindrical container.

If a tennis ball has a diameter of 6.6 centimeters, what is the approximate volume of the container?

A. 450 cubic cm
B. 677 cubic cm
C. 1203 cubic cm
D. 2708 cubic cm
20) The Rose Center for Earth and Space in New York City has a planetarium in the shape of a sphere. The sphere has a diameter of 26.5 meters. What is the volume of the sphere?
A. 735 cubic meters
B. 2205 cubic meters
C. 8820 cubic meters
D. 9739 cubic meters

21) This cone has a radius of 25 feet and a slant height of 65 feet.

What is the height of the cone?

22) Maya is building a lamp out of plastic. Their design is a rectangular prism with a triangular prism on top.


Part 1. What is the volume of the lamp?
A. 128 cu in
B. 384 cu in
C. 512 cu in
D. 640 cu in

Part 2. What is the surface area of the lamp? (Note that there is no plastic on the bottom of the cube.)
23) What is the total surface area of this flower pot?

A. 2,628.18 square inches
B. $2,882.52$ square inches
C. 3,391.2 square inches
D. 5,765.04 square inches
24) The diagram shows the dimensions of a fish tank. Josephine is filling the tank at a rate of 400 cubic inches of water per minute. Approximately how long will it take her to fill the tank to a height of 15 inches?

A. 11 minutes
B. 13 minutes
C. 15 minutes
D. 17 minutes
25) The glass sphere of a snow globe has a surface area of 63.585 square inches. What is the approximate diameter of the snow globe?
A. 2.25 inches
B. 4.5 inches
C. 6.75 inches
D. 20.25 inches


## Test Practice Questions - Answer Key

Note: The explanations given are not the only way to solve these problems. They are just an example of one way. If you did it differently, and got the correct answer, then you probably used an effective strategy.

1) Choice $C$. The solid is a cube, so all of the edges are equal in length. Since the volume is 125 cubic inches, calculating the cube root will give us the length of each edge.
$\sqrt[3]{125}=5$ inches
If each edge is 5 inches, then the area of 1 face of the cube is
 25 square inches. Since a cube has 6 faces, and each face is 25 square inches, then the cube will have a total surface area of 150 square inches.
2) Choice B. We can calculate the surface area of a sphere by multiplying the area of the great circle by 4 . Since the diameter of the sphere is 2.9 , the radius is $1.45 .1 .45^{2}=$ 2.1025. When we multiply $2.1025 \times 3.14=6.60185$. Then $6.60185 \times 4=26.4074$. If you chose $A$, you might have forgotten to multiply the area of the great circle by 4 . If you chose C , you might have used the diameter instead of using the radius.
3) Choice $B$
A. Surface Area $=312$ square cm
B. Surface Area $=340$ square cm
C. Surface Area $=292$ square cm
D. Surface Area $=232$ square cm
4) Choice $A$
A. Volume of cube $=\mathbf{2 1 6}$ cubic in
B. Volume of sphere $=113$ cubic in
C. Volume of cone $=56.5$ cubic in
D. Volume of cylinder $=169.6$ square cm
5) Choice $\mathbf{C}$.
A. Volume: 339 cubic feet
B. Volume: 1357 cubic feet
C. Volume: 1885 cubic feet
D. Volume: 1131 cubic feet
6) Choice D. The radius of a sphere that has a surface area of 1808.64 square inches is 12 inches. The volume of a sphere with a radius of 12 inches is $7,234.6$ cubic inches.

The formula for the surface area of a sphere is $S A=4 \pi r^{2}$

| $\mathbf{r}$ | $\mathbf{r}^{2}$ | $\pi r^{2}$ | $4 \pi r^{2}$ | SA |
| :---: | :---: | :---: | :---: | :---: |
| 12 | 144 | 452.16 | 1808.64 | 1808.64 |

Once we know the radius is 12 inches, we can use the formula for the volume of a sphere. $V=\frac{4}{3} \pi r^{3}$

| $\mathbf{r}$ | $\mathbf{r}^{3}$ | $\pi r^{3}$ | $\frac{4}{3} \pi r^{3}$ | $\mathbf{V}$ |
| :---: | :---: | :---: | :---: | :---: |
| 12 | 1728 | 5425.92 | 7234.56 | 7234.56 |

7) Choice A. The volume of each box of cereal is $225 \mathrm{in}^{3}$. We know the dimensions of two sides of the second cereal box. $6 \times 3 \times 12.5=225$.
8) Choice $B$. The surface area of the side of the tank is approximately 1809 square feet. $1809 \div 400$ is approximately 4.5 cans. Since they can't buy half a can, Adela needs to buy 5 cans of paint.
9) Choice C. The dimensions of the box are 6 inches by 4 inches by 15 inches and a volume of 360 cubic inches.
10) Choice B. Each tree needs 9.4 square feet of fencing, so 5 trees would require approximately 47.1 square feet of fencing. If you chose $D$, you might have used used the diameter of 1 foot instead of the radius of half a foot ( 0.5 or $1 / 2$ )
11) Choice $A$. The total volume is 900 cubic cm . The volume of the dark prism is 240 cubic cm . So the remaining prism has a volume of 660 cubic cm .
12) Part 1: Choice $C$.

Part 2: Choice C. The volume of one box is 468,000 cubic mm , so the volume of 9 boxes is $4,212,000$ cubic mm .
13) Choice A.
14) Part 1: Choice B.

## Part 2: Choice C.

15) Choice $\mathbf{C}$. The volume of the cylindrical part of the container is 3077.2 cubic feet. The volume of the cone part of the container is 615.44 cubic feet.
16) Choice $B$. The volume of the water cooler container is approximately 1413 cubic inches. The volume of the paper cup is approximately 9.42 cubic inches. $1413 \div 9.42=$ 150
17) Choice B. The volume of the smaller bed is 72 cubic feet and the volume of the larger bed is 360 cubic feet. 360 cubic feet -72 cubic feet $=288$ cubic feet.
18) Choice $A$.
19) Choice B. The diameter of the tennis ball is equal to the diameter of the cylinder. The diameter of a tennis ball is 6.6 cm , so the radius of the tennis ball is half of that, or 3.3 cm . Since 3 tennis balls fit in the container, the height of the cylinder is 6.6 cm multiplied by 3 , or 19.8. Choice $A$ is the total volume of the 3 tennis balls. If you choose choice $D$, you may have used the diameter of the cylinder and not the radius.
20) Choice D.
21) 60 feet. The radius, height, and slant length of a cone form a
 right triangle where the height and radius are the legs and the slant length is the hypotenuse. We are given the hypotenuse of 65 feet and a leg of 25 feet.

$$
\begin{gathered}
65^{2}=4225 \quad 25^{2}=625 \\
4225-625=3600 \\
\sqrt{3600}=60
\end{gathered}
$$

22) Part 1. Choice $\mathbf{C}$. The volume of the cube section is $6 \times 8 \times 8$ or 384 cu. inches. The volume of the triangular prism section is 128 cu . inches. Combined, the volume of the entire lamp is 384 $+128=512$.

Part 2. 298.4 square inches.
This net can help you visualize the surface area of the lamp.

23) Choice A. The flower pot is a cylinder. The surface area of the bottom is 254.34 square inches. The surface area of the curved side is $2,373.84$ square inches. 254.34 square inches $+2,373.84$ square inches $=2,628.18$ square inches. If you choose Choice $B$, you may have included the top of the cylinder in the formula, but there is no top to this cylinder.
24) Choice A. The volume of the tank filled to a height of 15 inches is $4200 \mathrm{in}^{3}$. We can divide 4200 cubic inches by 400 cubic inches per minute to calculate how long it would take to fill the tank. $4200 \div 400=10.5$ minutes. So 11 minutes is the best answer. Choice $B$ is how long it would take to fill the tank up to a height of 18 inches.
25) Choice B.

## The Language of Geometry

## Concept Circle

Explain these words and the connections you see between them.



## Fill in the Blanks

Use the words in the box below to fill in the blanks in the sentences below. Words may be used more than once.


Understanding surface area and volume is important when dealing with 3D solids in the real world. $\qquad$ are the shapes of most boxes or buildings. We calculate their
$\qquad$ by adding up the areas of all their faces. We calculate their volume by multiplying $\qquad$
$\qquad$ , and $\qquad$ .

We see $\qquad$ in structures like roofs or in the shape of a tent. We calculate their volume by finding the area of the $\qquad$ and multiplying by the
$\qquad$
$\qquad$ are a special type of prism, with all equal sides. One way to find the surface area is to square 1 side and multiply by $\qquad$ . We can calculate the volume by $\qquad$ the length of a side. $\qquad$ and
$\qquad$ are similar because they both have slanted sides that come together to form an $\qquad$ We see $\qquad$ in the shapes of drops of water and balls. The formulas for determining surface area and volume of this solid rely on the
$\qquad$ , which is half of the $\qquad$ We see $\qquad$ , in objects like cans or pipes. The formulas for calculating surface area and volume of this solid use the radius and the $\qquad$ .

## Go Forth and Measure

The world is full of three-dimensional objects that you now know how to measure.
Find a real-life example of a three-dimensional object. Measure the dimensions of the object with a ruler. Then do your best to calculate their surface area and volume. Add what you learn to the chart below. If you don't have a ruler, don't worry - you can use this page to estimate! The long side of this piece of paper is 11 inches and the short side is $81 / 2$ inches.

Can you find an example of each: Rectangular Prism, Cube, Triangular Prism, Pyramid, Cylinder, Cone, Sphere.

| 3D Figure | Real-Life <br> Object | Dimensions | Surface Area | Volume |
| :---: | :---: | :---: | :---: | :---: |
| Rectangular <br> Prism | Cereal box | $11^{\prime \prime} \times 6^{\prime \prime} \times 3^{\prime \prime}$ | 234 sq inches | 198 cubic inches |
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3D Models


These models were constructed with toothpicks and raisins. (Marshmallows work well too.)
How many three-dimensional solids can you construct?

What questions do you still have about three-dimensional geometry?

|  |
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## Vocabulary Review

## Graphic Organizers

In order to learn vocabulary, we need practice using it in different ways. Choose words that you want to practice. Use the graphic organizer on the next page as a model to create as many graphic organizers as you need on paper or in a notebook.

To complete the graphic organizer, you will choose a word and then answer four questions:

- What is the definition of the word? You can look at the glossary and the vocabulary review at the end of the packet for help. Try to write the definition in your own words.
- Make a visual representation. You can make a drawing or diagram that will help you remember what the word means.
- What are some examples of the word you're studying?
- What are some non-examples of this word? These are things that are not examples of the word you're studying.

Look at the sample for the word quotient below.

| What is it? |
| :--- |
| A quotient is the result of |
| dividing one number by another. |
| It is the answer to a division |
| question. |
| What are some examples? <br> 15 divided by 3 equals $\mathbf{5}$ <br> $66 \div 6=11$ <br> $63 / 18=3.5$ |
| 5, 11 and 3.5 are quotients in <br> these calculations. <br> population $\div$ are $=$ density |

Three-Dimensional Geometry (Part 1)


## Glossary

You can use this section to look up words used in this math packet.
approximate (adjective): close to the actual, but not completely accurate or exact. We can
 apex (noun): the point opposite the base in cones and pyramids.
attribute (noun): An attribute is an aspect of an object that can be measured. Other words that have a similar meaning are: quality, trait, characteristic, feature, property.
base (noun): The base of a three-dimensional object is the surface the object stands on.
circle (noun): a two-dimensional figure in which every point is the same distance from a point called the center. See Circumference, Radius, Diameter.
circular (adjective): We use the word circular to describe an object that looks like a circle. For example, a wall clock can be circular.
circumference (noun): The distance around a circle. You may think of the circumference as the perimeter of a circle. The circumference of every circle is
 a little more than three times the length of the diameter of the circle. To be more exact, the circumference of a circle is equal to pi ( $\pi$ ) times the length of the diameter. In the circle above, the circumference is the distance from point C all the way around the circle back to point $C$. That distance is a little more than three times the length of line $B C$.
compare (verb): to describe how two or more things are alike or different.
cone (noun): A 3-dimensional object that has a circular base joined to a point by a curved side. The point is called an apex.
conical (adjective): We use the word conical to describe an object that looks like a cone. For example, a traffic cone is conical.
cube (noun): A box-shaped solid object that has six identical square faces.
cube (verb): To cube a number means to multiply that number by itself and then by itself again. For example, 12 cubed means $12 \times 12 \times 12.12$ cubed can also be written as $12^{3}$.
cylinder (noun): A solid object that has two identical flat ends that are circular and one curved side.
cylindrical (adjective): We use the word cylindrical to describe an object that looks like a cylinder. For example, a tree trunk is cylindrical.
cube root (noun): The cube root of a number is a value that, when multiplied by itself and then multiplied by itself again, gives the number. The cube root of 8 is 2 .
depth (noun): the distance from the front to the back of a solid.
diameter (noun): a line segment that goes from one point on a circle, through the center of the circle, to another point on the circle. The diameter is equal to two times the radius. See Circle, Radius
dimensions (noun): A dimension is a measurement of length in one direction. Height, width, depth are all examples of dimensions. A line has one dimension (1D), a square has two dimensions (2D), and a cube has three dimensions (3D).

edge (noun): A line segment between faces. Rectangular prisms and cubes have 12 edges. Triangular prisms have 9 edges. A square pyramid has 8 edges. Cylinders, cones, and spheres have no edges.
exponent (noun): an exponent tells us how many times to multiply a number by itself.
Exponents are written next to the number, but raised. For example, $8^{3}$. $8^{3}$ means $8 \times 8 \times 8$. 3 is the exponent that tells us to multiply 8 by itself three times.
face (noun): A flat surface on a three-dimensional object. Rectangular prisms have 6 faces. Triangular prisms have 5 faces. Square pyramids have 5 faces. Cylinders have 2 faces. Cones have 1 face.
height (noun): the distance from the bottom to the top of a 3D solid standing upright. The height can also be understood as the distance between the two flat ends of a prism or a cylinder. Or the the shortest distance from the vertex of a pyramid or cone to the base.
hemisphere (noun): a hemisphere is exactly half of a sphere. For example, this orange has been cut into two hemispheres.

length (noun): The measurement of something from one end to the other. The distance between 2 points. Height, base, depth, width, side, perimeter are words used to describe length.
net (noun): a pattern you can cut and fold to make a model of a 3D solid. For example:
This is a net of a cube. This is a net of a cylinder.

oblique prism (noun): An oblique prism is a prism that seems like it is tilting. The bases do not appear directly above one another when an oblique prism is sitting on its base.
perimeter (noun): The distance around a two-dimensional shape or area. The perimeter of a circle is called a circumference.
pi ( $\pi$ ): Pi is the 16th letter in the Greek alphabet. It is used in math to represent a special relationship found in every circle. The circumference of any circle in the world is a little more than 3 times greater than the diameter of that circle. If you divide the length around any circle
(circumference) by the length across it (diameter), you will get something close to 3.14 . See Circle, Circumference, Diameter.
prism (noun): a solid object with two identical ends and flat sides. The shape of the two identical ends give the prism its name. For example, a triangular prism has a triangle at each end.
pyramid (noun): a solid with base is a flat shape (like a square) and the sides are triangles that meet at a point at the top of the solid.
radical (noun): A symbol that means "root." Radicals are used for square roots, cube roots, and other roots. The radical for cube roots is $\sqrt[3]{ }$
radius (noun): a line segment from the center of a circle to any point on the circle. The radius is half of the diameter. Every radii in a circle is the same length. In a sphere, the radius is the distance from anywhere on the surface of the sphere to the center of the sphere. The way to describe more than one radius is "radi"" (pronounced ray-dee-i).
rectangular prism (noun): A solid (3-dimensional) object which has six faces that are rectangles.
right prism (noun): A right prism is a prism where one base appears directly above the other base. All of the prisms you worked with in this packet are right prisms.
slant height (noun): The distance up the side, going from the base up to the apex of a pyramid or cone. Also called slant length. In the diagrams to the right, both slant heights are labeled $s$.

solid (noun): A three dimensional (3D) object. Examples of solids include rectangular prisms, spheres, cubes, pyramids and cylinders.
sphere (noun): a three-dimensional object shaped like a ball. Every point on the surface of the sphere is the same distance from the center of the sphere.
surface area (noun): the total area of the surface of a three-dimensional object. When we talk about surface area, we are talking about how many square units it would take to cover a surface.
triangular (adjective): We use the word triangular to describe an object that looks like a triangle or triangular prism. For example, many rooftops are triangular.
triangular prism (noun): a prism with a triangle at each of its ends. A triangular prism has 2 faces that are triangles and 3 faces that are rectangles.
unit (noun): a quantity used to measure other quantities

- Inches, feet, centimeters and meters are units for measuring distance or length.
- Square inches, square feet, square centimeters, and square meters are units for measuring area.
- Cubic inches, cubic feet, cubic centimeters, and cubic meters are units for measuring volume.
vertex (noun): A point where two or more line segments meet. A cube has eight vertices (plural of vertex).
volume (noun): A measurement of the 3-dimensional space something takes up. When we talk about volume, we are talking about how many cubic units it would take to fill that space.
width (noun): the distance from side to side of a solid.


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## Version

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