Problem-Solving in Transportation and Warehousing: Packing a Shipping Container

3 HOURS

Students work in groups to solve a real-world math problem of determining how to arrange boxes on a pallet to fit the maximum number of boxes.

NOTE: *One of the standards assessed by the TASC exam requires students to solve real-world mathematical problems involving volume of three-dimensional objects composed of right prisms. This problem allows students to apply geometric methods to a real-world situation. Students at multiple levels can contribute and develop mathematical understanding. It is helpful but not essential for students to have been previously introduced to the concepts of length, area and volume. In the course of the lesson, they will understand that volume is a measurement of the space inside an object, in this case a cardboard box and the 3-dimensional space on a pallet.*

NUMERACY TIE-IN
- Estimation
- Geometry
- Volume
- Using math to support a claim

PREP

- Review the handouts and solve the pallet problem.
- Consider strategies your students might use.
- Optional: Cut out *Nets 1, 2 and 3.*

MATERIALS

- Optional and recommended: calculators
- Optional and recommended: rulers or tape measures
- Optional: a few empty cardboard boxes
- Optional: 1 cm cubes, available from *ETA Hand2Mind*
- Handouts:
  - *Length, Area, and Volume*
  - *Finding the Area of Nets*
  - *Two Stacks*
  - *Which Stack is Bigger?*
  - *Packing Boxes on a Pallet*
  - *Converting Feet and Inches*
  - *Shipping Pallets of Boxes*
  - *Grid paper*
  - *Scissors and Tape*

PART 1: Introduction to Volume and Surface Area

EXPLAIN

1. **Ask:** In what situations have you used measurements in inches, in feet, in square inches, in square feet, in cubic inches and cubic feet? Discuss with a partner and write down your answers.

   › *Examples of possible responses include:*

   Inches: *my height, length of my foot, length and width of a piece of paper, inseam of pants, length of screws or nails.*

   Feet: *my height, length and width of a room, length of a swimming pool, distance around a running track.*

   Square inches: *amount of paper needed to wrap a gift, small tiles needed for home construction projects.*

   Square feet: *paint coverage, yard space, tiles on a kitchen floor, size of an apartment or house.*

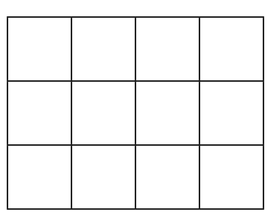
   Cubic inches: *size of a car motor.*

   Cubic yards: *buying mulch or gravel, how much an 18-wheeler holds, space in a closet.*

2. Distribute and discuss the *Length, Area and Volume* handout.

   NOTE: In order to understand *surface area*, your students will need to understand *area*. When we measure area, we are determining the number of squares of a certain size that cover a flat, 2-dimensional space. In reality, there is no difference between the two concepts. Surface area is just the idea of area applied to the surface of a 3-dimensional shape. You may want to review this sequence with students:

3. Draw or display this array on the board.



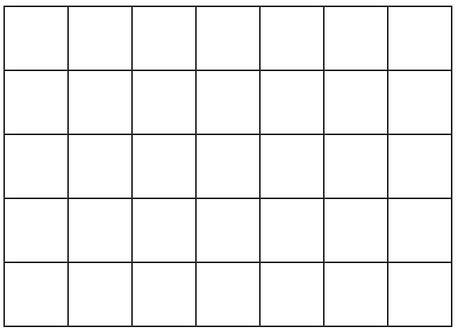
Say: We are going to look at two concepts today: Area and Volume. Area is a measurement of how many squares cover a flat space. When you look at this grid, how many squares do you see? In other words, what is its area?

› *12.*

How do you know?

- *There are 3 rows of 4 squares.*
- *There are 4 columns of 3 squares.*
- *3 times 4 is 12.*
- *I counted them.*

3. Draw this array.



What is the area of this rectangle?

- *35.*

How do you know?

- *5 × 7, 7 × 5, 5 rows of 7*

Would I get the same number if I counted all the squares?

- *Yes.*

That would absolutely tell me the area of this figure. It might take a while, but it's a perfectly legitimate way to find the area of a rectangle. It's important that you remember that finding the area is about counting squares. However, you might have faster, more efficient ways of finding the number of squares that don't require you to count every square.

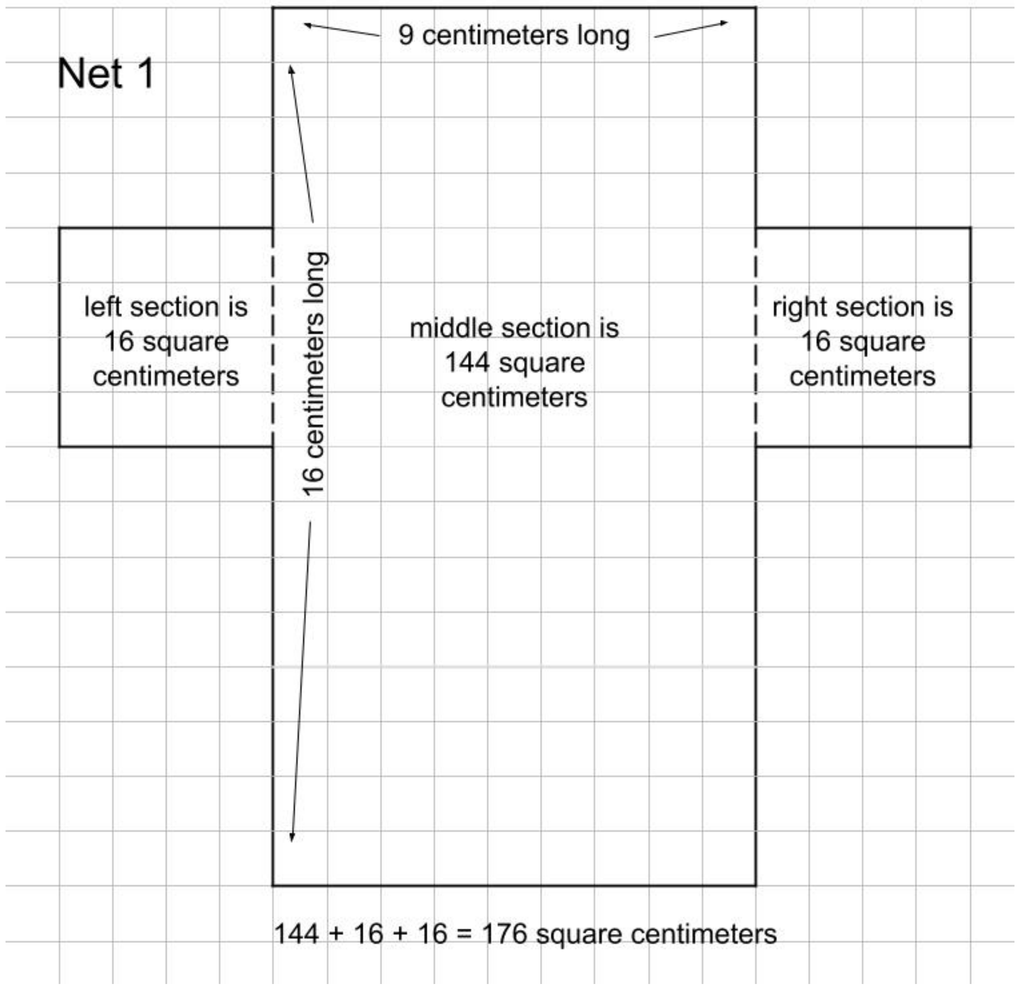
4. Distribute *Finding the Area of Nets*. If you cut them out in advance, it will make folding faster later on, but students can also cut out the nets later. Ask students to find the area of the three figures.

5. **Say:** Share how you found the area of the different figures.

There are many different ways of arriving at the same answer. If possible, look for students who found the area in a few different ways, so that students can see that there isn't just one way to solve a problem.

NOTE
These figures are called nets. A net is a 2-dimensional pattern for making a 3-dimensional shape. Students will use these nets later to make boxes (rectangular prisms).

Example of one way of seeing the area of Net 1:



Net 1: Area of 176 square centimeters

Net 2: Area of 174 square centimeters

Net 3: Area of 166 square centimeters

For more information on the instructional routine Notice/ Wonder, visit **http://bit.ly/CollectEdNYNotWon**

6. Look at a handout or projected image of the first page of the *Two Stacks* handout. Write down at least two things you notice about the two stacks, then write at least two questions.

7. Share your noticings and wonderings with a partner. Write down things your partner noticed and wondered, then come up with other possible questions as a pair. In a large group discussion afterwards, take notes so that students can see what everyone noticed and wondered.

Students may notice:

- *Each stack is made of cubes of the same size.*
- *The cubes are different colors.*
- *The stacks have different measurements.*
- *Both stacks have 5 layers.*
- *Stack A is wider.*
- *Stack B is longer.*

Make sure that students see:

- Stack A is 4 cubes wide, 7 cubes long and 5 cubes high.
- Stack B is 3 cubes wide, 9 cubes long and 5 cubes high.

Questions to prompt responses could include:

- How many cubes are there in each stack?
- Which stack has more cubes?
- Is there a way to find out how many cubes there are without counting each one?
- How big is each cube?
- Why did someone make these stacks?

8 As mathematicians we refer to these stacks of cubes as **rectangular prisms**, because they are solid (3-dimensional) objects with six faces that are rectangles. Rectangular prisms are also sometimes called cuboids (cube-like). Cubes are a special kind of rectangular prisms where each dimension is the same length. All cubes are rectangular prisms, but not all rectangular prisms are cubes. This is because squares are rectangles, so each face of a cube is a rectangle. Are you surprised to hear that a square is a rectangle? This is because the basic definition of a rectangle is that opposite sides are equal and there are four 90-degree angles? Check to see if a square meets these conditions. Not all rectangles are squares, but all squares are rectangles.

9 In mathematics, a **net** is a pattern that you can cut and fold to make a model of a solid shape. One common example of a net is an unfolded cardboard box. If you have worked in a pizza shop, you may have had to fold pizza boxes. When pizza boxes are delivered to the shop, before they are folded, they would be considered nets. Can you think of other examples of nets?

Why do you think these diagrams are called nets?

> *Maybe it's because they wrap around solid 3-dimensional figures like a net.*

10 Distribute scissors and tape.

11 Cut out the nets and fold them into rectangular prisms, attaching the edges with tape.

12 Complete the second page of the *Two Stacks* handout.

13 After discussing student solutions, you should make the following points:

- We measure area by counting how many squares would fit in a space. Wrapping paper around an object is a measure of surface area. Surface area is the same as other kinds of area, except that it generally refers to area on the outside of an object like a rectangular prism, cylinder or pyramid. A question like, **How much paper would it take to wrap a gift?**, is about surface area. We find the surface area of a 3-dimensional object by thinking about how many squares would cover its surface. Squares are flat, 2-dimensional objects, as opposed to cubes, which are 3-dimensional.
- In this case we're measuring these rectangular prisms with square centimeters. If we were measuring the surface area of a wall, we might use square feet. **What if we want to measure the surface area of Earth, what measurement do you think we would use?** When measuring surface area, we can use squares of different measurements, such as square meters, square inches, square feet, square miles, etc. If we're talking about area, then we use squares.
- When we write a measurement of area, we always include the unit to show the size of the squares we're using to measure the space. In this case, our units are *square centimeters*, which we can also write as cm^2. The ^{2} is an exponent that means we are talking about squares, which are measured with *length* and *width*, two measurements that are multiplied by each other.
- To find the number of squares (surface area) on the outside of a rectangular prism, you can use this formula:

$$SA = 2lw + 2lh + 2hw$$

SA stands for surface area, l for length, w for width, h for height. Students may not realize that each letter represents a measurement from the rectangular prism. They might also not realize that, in a formula, a number next to a letter or a letter next to a letter means that the two quantities should be multiplied so that $2lw$ means $2 \times l \times w$.

You can explain the formula for surface area of a rectangular prism by referring to the earlier solution: $27 + 27 + 45 + 45 + 15 + 15 = 174$. Do students see any similarities between the formula and this solution? By grouping the areas of opposite sides of the prism, students may see the surface area formula in this calculation: $2(3 \times 9) + 2(5 \times 9) + 2(3 \times 5) = 174$.

Ask students:

Which of these two stacks is bigger?

The answer depends on what kind of measurement you're using (height, width, length, volume surface area).

14 Distribute and discuss the *Which Stack is Bigger?* handout.

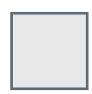
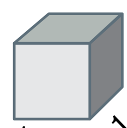
Length, Area, and Volume

When we measure the size of objects, length, area, and volume are three important measurements that we use to understand the size of things in the world.

LENGTH is the distance between two points. This could be a short distance such as an inch or centimeter or a large distance like a mile or kilometer. Your height is a measure of length. We measure distance with rulers, tape measures and odometers (in a car). Height, width, and depth are all measures of length because they are measuring the distance between two points.

AREA is the size of a surface. We measure area by counting the number of squares that would cover a surface. These could be square inches, square centimeters, square yards, square miles, etc. The amount of wall space covered by a poster is a measure of area.

VOLUME is the amount of space that an object fills. When we measure volume, we imagine filling an object with cubes and then count how many cubes will fit inside the object. You can think about this as the number of cubes that would fill the same amount of space as the object. The amount of water in a swimming pool is a measure of volume.

	Length	Area	Volume
Drawing	1 cm	 1 cm 1 cm	1 cm  1 cm 1 cm
Ways of writing	1 centimeter 1 cm	1 square centimeter 1 cm^2	1 cubic centimeter 1 cm^3
Ways of saying	"1 linear centimeter"	"1 square centimeter"	"1 cubic centimeter"
Dimensions	length	length and width	length, width and height

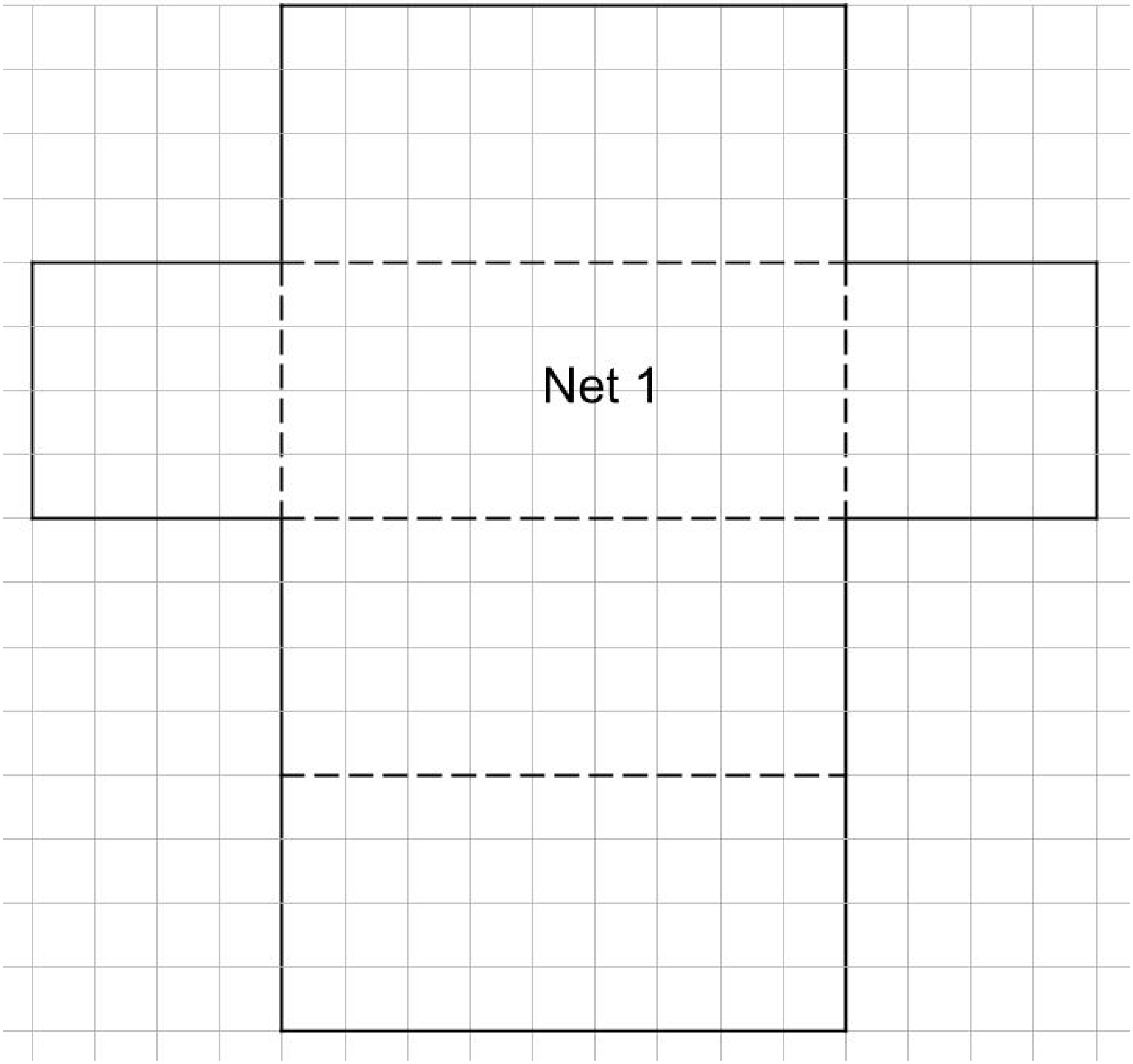
Look around you. Choose an object to describe.

What part of the object is its length?

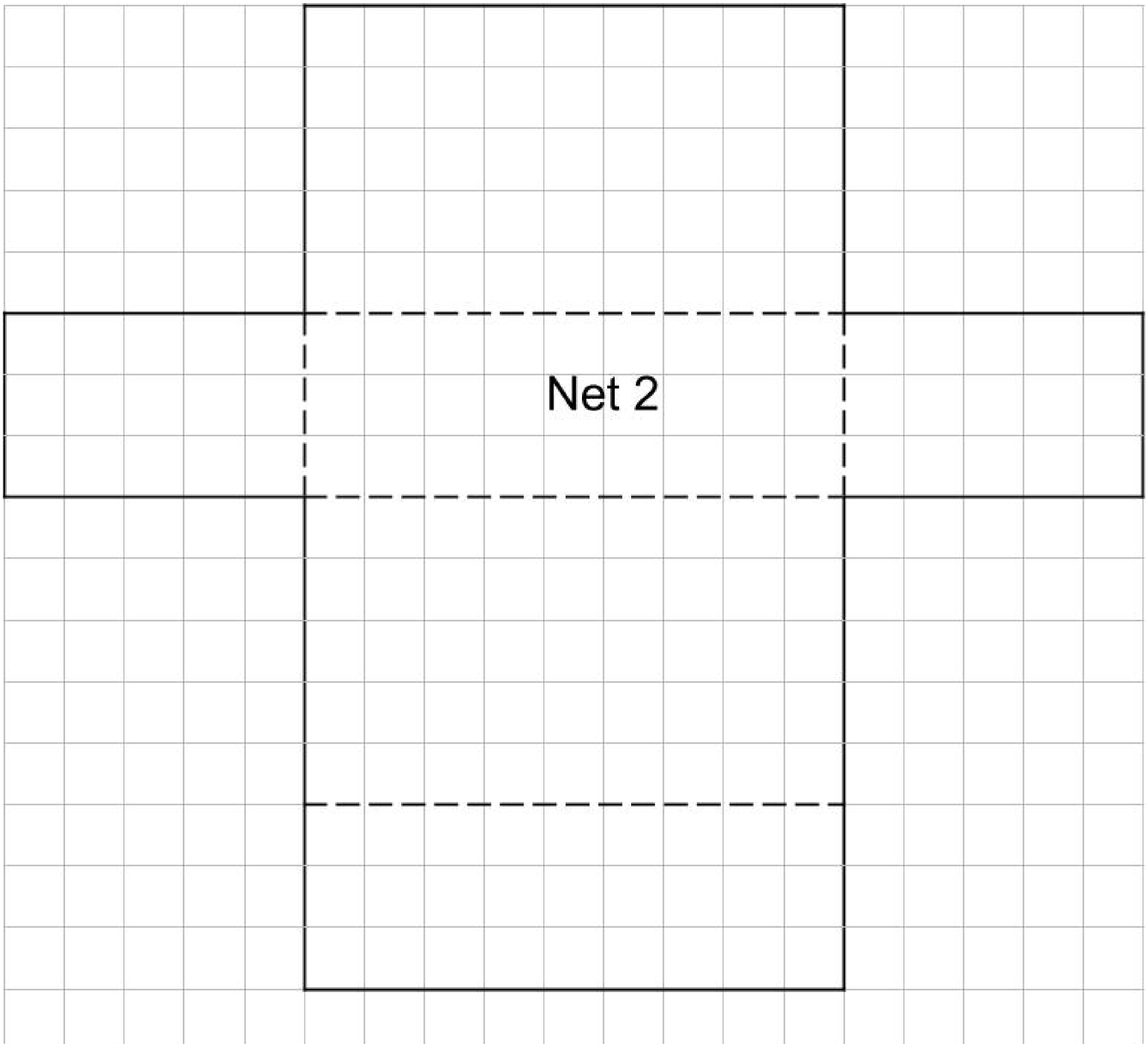
What part of the object is its area?

What part of the object is its volume?

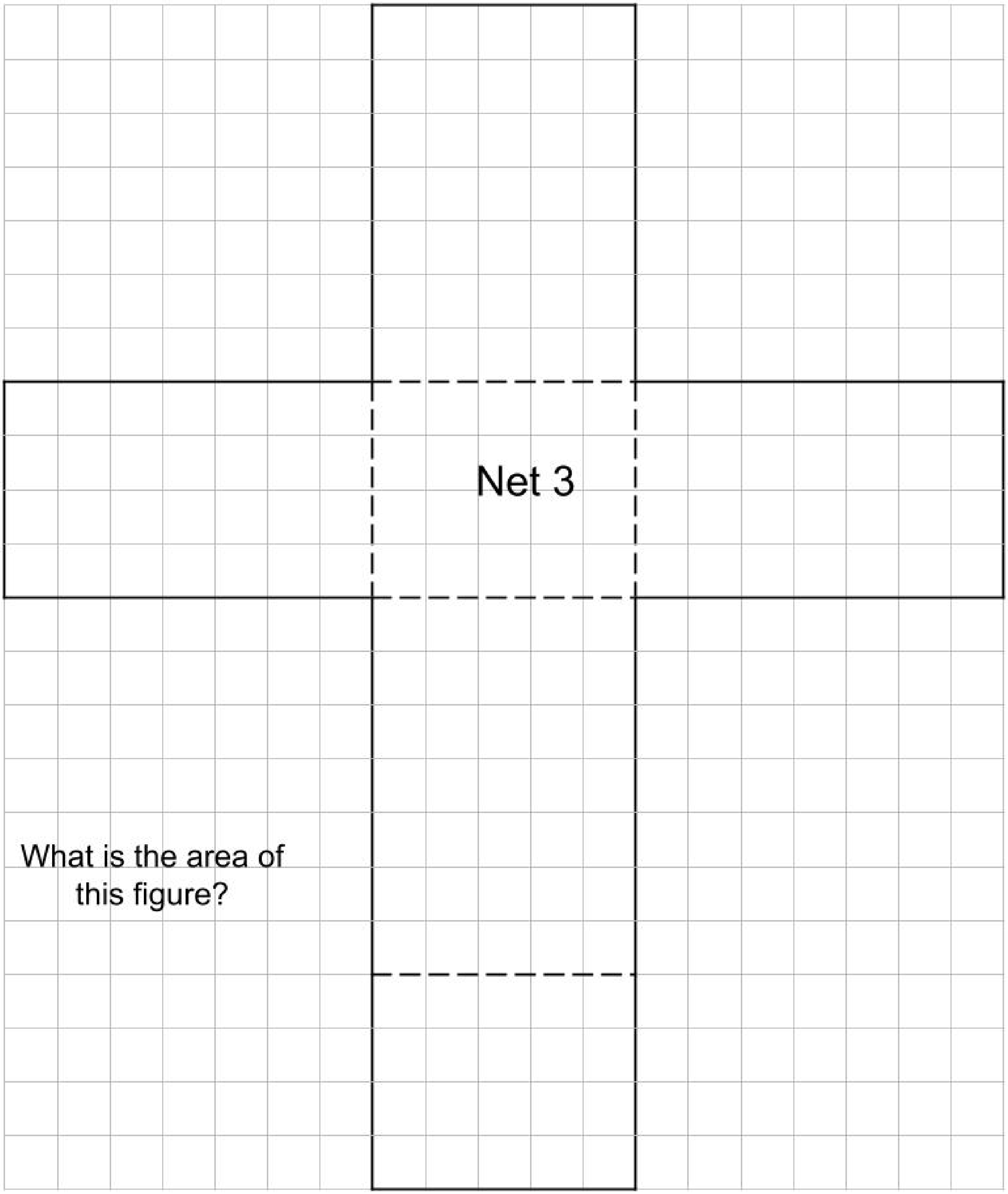
Finding the Area of Nets



1 **NET 1: What is the area of this figure?** ______________________________

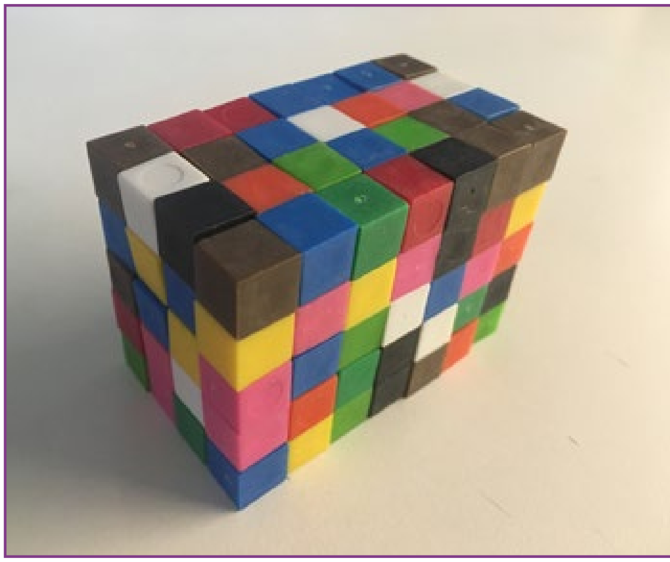


2 **NET 2:** What is the area of this figure? ______________________

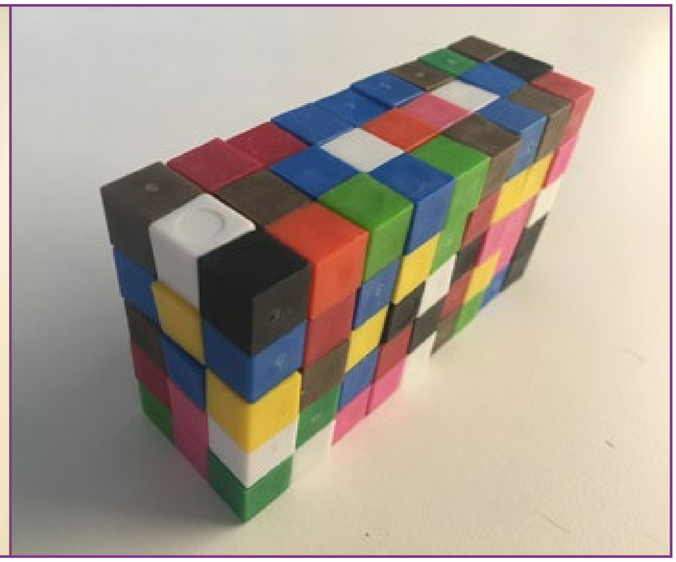


3 **NET 3:** What is the area of this figure? ____________________

Two Stacks



STACK A



STACK B

1. What do you notice?

2. What do you wonder?

3. Which of Nets 1, 2, and 3 is a model for Stack A? How do you know?

4. Which of Nets 1, 2, and 3 is a model for Stack B? How do you know?

5. If you made Stack A and Stack B with 1cm cubes, which stack would require more cubes to build? Explain your answer below with words, numbers and/or drawings.

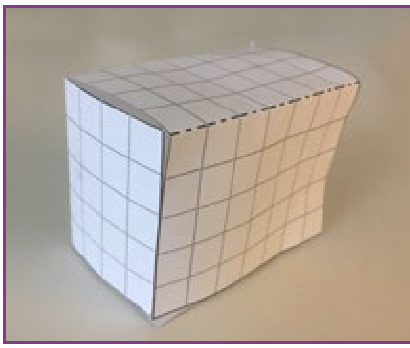
6. If you were to cover Stack A and Stack B with paper, which would require more paper? Explain your answer below with words, numbers and/or drawings.

Teacher Notes and Answer Guide for Two Stacks

The following notes are included for teacher support of Two Stacks.

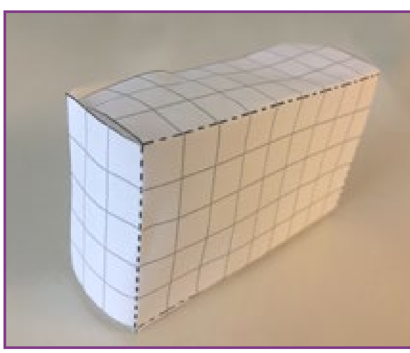
Which of these nets is a model for Stack A? How do you know?

Net 3 because if you measure each of the sides, you will see that each rectangular face has the same dimensions as Stack A: 4cm by 7cm on top, 5cm by 4cm on the front and 5cm by 7cm on the side.




Which of these nets is a model for Stack B? How do you know?

Net 2 because if you measure each of the sides, you will see that each rectangular face has the same dimensions as Stack B: 3cm by 9cm on top, 5cm by 3cm on the front and 5cm by 9cm on the side.




If you made these two stacks with 1cm cubes, which stack would require more cubes to build? Explain your answer below with words, numbers and/or drawings.

Emphasize conceptual understanding over memorizing formulas

The goal of this activity is not for students to use the formula $l \times w \times h$, but for students to understand that volume is less about a formula and more about calculating the number of cubic units in an object. At this stage of the lesson, it is better to avoid using the formula for volume and instead focus on the number of cubes. **Avoid telling students the formula for volume**. There many other ways that students can find the number of cubes which will support their understanding better than using the formula. Later, after they understand that volume is about counting cubes and can explain why this involves multiplication, it will be appropriate for them to use the formula.

Here are some possible solutions:

- Counting the number of cubes in a horizontal layer and adding the layers together. For example, Stack A has 28 cubes in each horizontal layer. There are 5 layers, so 28 + 28 + 28 + 28 + 28 = 140 cubes. Students may find the number of cubes in each layer by multiplying (4 × 7) and then multiplying by the number of layers (5).
- Counting the number of cubes in each vertical layer and adding the layers together. For example, Stack A has 35 cubes in each vertical layer. There are 4 layers, so 35 + 35 + 35 + 35 = 140 cubes. Students may find the number of cubes in each vertical layer by multiplying (5 × 7) and then multiplying by the number of layers (4).
- *Length* × *width* × *height*. Some students will remember the formula for volume and will correctly apply it to find the number of cubes in each stack. If a student quickly comes up with 140 cubes for A and 135 cubes for B, ask if they can prove with a drawing why their answer is true. (In discussions with students later, this should be the last strategy to discuss. It is better to talk about alternative strategies students may have come up with, such as the two described above.)

Discussing process

When students can articulate the methods they used to solve a problem, it helps to solidify their understanding of it, and helps other students find multiple ways of solving the problem. You might do this by having students present how they figured out the answer, writing steps on the board.

After discussing solutions, you should make the following points:

- The number of cubes in each stack is a measure of volume. When we measure volume, we're trying to figure out the size of 3-dimensional space, like a box or a room. We measure volume by counting how many cubes would fit in a space. In this case we're measuring with cubic centimeters.

If we were measuring the amount of air in this room, we might use cubic feet because that would be easier to count than all these little cubic centimeters. If we were talking about the volume of Earth, we might talk about cubic miles, since it's such a huge space. **Can you think of examples of other cubic measurements of volume?** Other measures of volume include cubic meters, cubic inches, cubic yards, etc. The important thing to remember is that if we use the word volume, we mean how many cubes it takes to fill the inside of an object using cubes.

- When we write a measurement of volume, we always include the unit to show the size of the cubes we're using to measure the space. In this case, our units are *cubic centimeters*, which we can also write as cm^3. The ^{3} is an **exponent** that means we are talking about cubes, which are measured with *length*, *width*, and *height*, three measurements that are multiplied by each other.
- If no one used $length \times width \times height$, you can give the following explanation: So we saw there are different ways to add up all the cubes. Another way is to calculate the number of cubes in one layer ($l \times w$) and then multiply that by the total number of layers ($\times h$).
- To find the number of cubes (volume) in a rectangular prism, as we saw earlier, you can multiply the length by the width by the height. This method is shown by the formula:

$$V = l \times w \times h$$

V stands for volume, l for length, w for width, h for height.

If I wanted to wrap all sides of these two stacks with paper, which would require more paper?

Similar to the volume question, the goal of this activity is not for students to use the formula for the surface area of a rectangular prism to determine how much paper is needed. They should be able to determine this by counting squares on the outside of the figure. It is best to avoid using the words *surface area* until after students already have an answer to which requires more paper. You can then connect the understanding they have developed to the definition of surface area and the formula, after you go over their solutions to the question.

Possible solutions include:

- Counting all the squares within the outline of each net. Net 3 has 174 squares. Net 1 has 166 squares. All the squares are the same size, therefore Net 3 uses more paper.

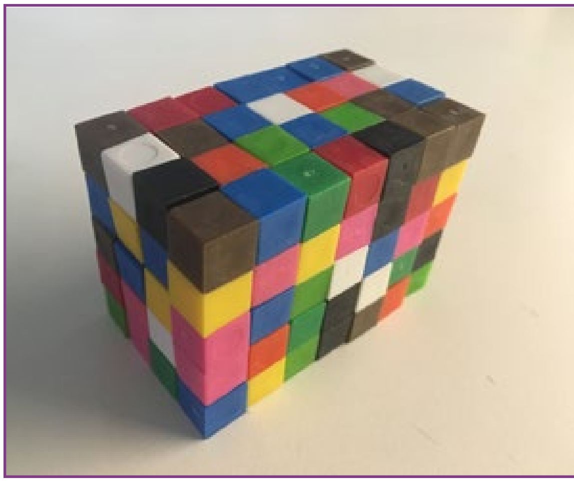
- Finding the area of each rectangular face of the prism using multiplication.

  Stack A: The top rectangle in has 28 squares because it is 4 cm wide and 7 cm long. The bottom rectangle is also 28 squares. One side of the prism is a 5 by 7 rectangle with 35 squares. The other side also has 35 squares. The rectangular side facing the camera is 4 by 5 with 20 squares. The opposite side is also made up of 20 squares. You can calculate the total number of squares by multiplying the length and width of each rectangle and adding them together: 28 + 28 + 35 + 35 + 20 + 20 = 166.

  Stack B: The top rectangle in has 27 squares because it is 3 cm wide and 9 cm long. The bottom rectangle is also 27 squares. One side of the prism is a 5 by 9 rectangle with 45 squares. The other side also has 45 squares. The rectangular side facing the camera is 3 by 5 with 15 squares. The opposite side is also made up of 15 squares. You can calculate the total number of squares by multiplying the length and width of each rectangle and adding them together: 27 + 27 + 45 + 45 + 15 + 15 = 174.

- Some students may know the formula for the surface area of a rectangular prism. If so, ask if they can prove that their answer is correct using the net of the rectangular prism.

Which Stack is Bigger?



STACK A **STACK B**

Distance: One way to measure these stacks is figure out their length, width and height. These are measurements of distance, which we can measure with a ruler. The stacks are made of 1-centimeter (cm) cubes. We measure distance by finding out how far it is from one point to another point.

Stack A is 4 cm wide, 5 cm tall and 7 cm long.
Stack B is 3 cm wide, 5 cm tall and 9 cm long.

Surface area: How big is the total surface of the object? In other words, how much paper would be required to cover it completely? Area is measured in squares. Since we're using centimeters to measure the prisms, we would record the area in square centimeters (cm^2). To find the surface area of a rectangular prism, you can use this formula:

$$SA = (2 \times l \times w) + (2 \times l \times h) + (2 \times h \times w)$$

which can be rewritten as... $SA = 2lw + 2lh + 2hw$

Stack A has a surface area of 166 cm^2.
Stack B has a surface area of 174 cm^2.
Stack B has a bigger surface area. In other words, it would require more paper to be covered.

Volume: Another way to measure the size of a rectangular prism is with volume. With this measurement, we're figuring out how many cubes would fill the prism. Since we're using centimeters, we would record the volume in cubic centimeters (cm^3). To find the volume of a rectangular prism, you can use this formula:

$$V = l \times w \times h$$

Stack A has a volume of 140 cm^3.
Stack B has a volume of 135 cm^3.
Stack A has a bigger volume. In other words, there are more cubes in Stack A.

PART 2: Stacking Boxes on a Pallet

1. Distribute *Volume and Surface Area Practice*, and ask students to complete it. This is a review of Part 1, and a preparation for the Packing Boxes on a Pallet problem. It's not necessary that every student have their own net. You might set up groups of 4, so that each student can build one box. Check to make sure that students remember how to calculate volume and surface area, and include the appropriate units when they record their answers.

2. Distribute scissors and tape so that students can create boxes from Nets A, B, C, and D.

   NOTE: The grid lines represent inches, but the nets are not to scale since the actual measurements wouldn't fit on an 8 1/2" x 11" piece of paper. To help students understand, you might ask students the longest measurement in Box A. One of the dimensions is 16 inches. This is bigger than a piece of paper, which explains why the net was scaled down to fit on a sheet that could be given as a handout.

3. Distribute *Packing Boxes on a Pallet*. Ask students to take a few minutes to read the first page a few times and highlight important information. Next, read the page aloud to the group and ask students to talk with a partner to see if they highlighted the same information. Ask for volunteers to share information that they highlighted. This might include:

   - Boxes are stacked on the pallet for shipping.
   - Companies try to fit as many boxes on the pallet as possible.
   - The stack can't be taller than 5 feet high.
   - The pallet is 4 feet long and $3^1/_3$ feet wide.
   - The pallet is 6 inches thick.

   If it doesn't come up, **ask students what 6" means**? **And what does 4' mean?** Make sure students understand that the single hash mark means feet and the double hash mark means inches.

   ' means **feet**
   " means **inches**

   Ask students, **How many inches are in 4 feet? How long is the pallet in inches? How do you know?** Then ask the same questions about $3^1/_3$ feet. Ask students to write the length and width of the pallet in inches.

4. Read the second page of the handout aloud. **What new important information is included?** Add to the list on the board.

   - Boxes come in different sizes.
   - The sizes of each box are given in inches.

   Do any of your students recognize that the dimensions of these boxes are the same as the boxes they created?

5 Students may have pointed out by now that there is no question on the handout. Ask them to take 2 minutes to write down a mathematical question based on the information they have been given. They should then share their question with a neighbor and then write down any other questions that come to mind.

6 Ask students to share their questions and write them on the board so that students can read each other's questions. After all the questions have been shared and written on the board, tell students that they have all raised interesting questions, but the one you'd like help answering is:

Which box would you use for packing a pallet and how would you pack the pallet?

Possible process for students to work on the question:

- Ask students to work independently on the problem for 5 minutes, explaining that they will continue their work in a group next. After 5 minutes, ask students to talk to each other about what they have done so far.
- Share a piece of newsprint with each group so they can prepare a poster that expresses their answer and explanation for which are the best boxes for a pallet. Ask them to include words, pictures, anything that will make it clear enough for a team of Packers to follow. Once the posters are done, ask students to put them on the wall for the group to discuss.

When discussing students' answers to *Packing Boxes on a Pallet*, you might ask these follow-up questions either while the group is still working or to the whole class afterwards:

- **Which box did this group use?**
- **What is the volume of this box?**
- **How many boxes were you able to fit onto the pallet while remaining within the safety recommendations?**
- **What is the total volume of the boxes you put on the pallet? This is the total volume of goods that you are able to ship with this box and packing arrangement.**
- **How did you arrange the boxes on the pallet? Can you make a drawing to help us understand your method?**
- **What is the maximum volume of goods that can be shipped with on a pallet?**

7. The handout *Converting Feet and Inches* for additional practice as an in-class assignment or homework.

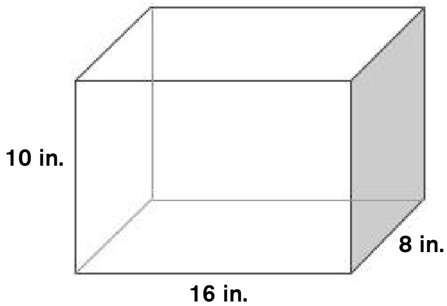
8. The handout *Shipping Pallets of Boxes* is included as an optional extension or homework activity.

> **Solution:** It is possible to arrange the 12" × 16" × 18" box so that 10 boxes fit in 3 layers for a total of 30 boxes, using the maximum space on the pallet. If boxes are packed in this way, 10 pallets would be required to ship 300 boxes.

Volume and Surface Area Practice

INSTRUCTIONS: Using scissors and tape, create boxes using Nets A, B, C, and D. Then calculate the volume and surface area. You might find it helpful to draw and label each box.

BOX A



Volume: ________ cubic inches

Surface Area: ________ square inches

BOX B

Volume: ________ in^3

Surface Area: ________ in^2

BOX C

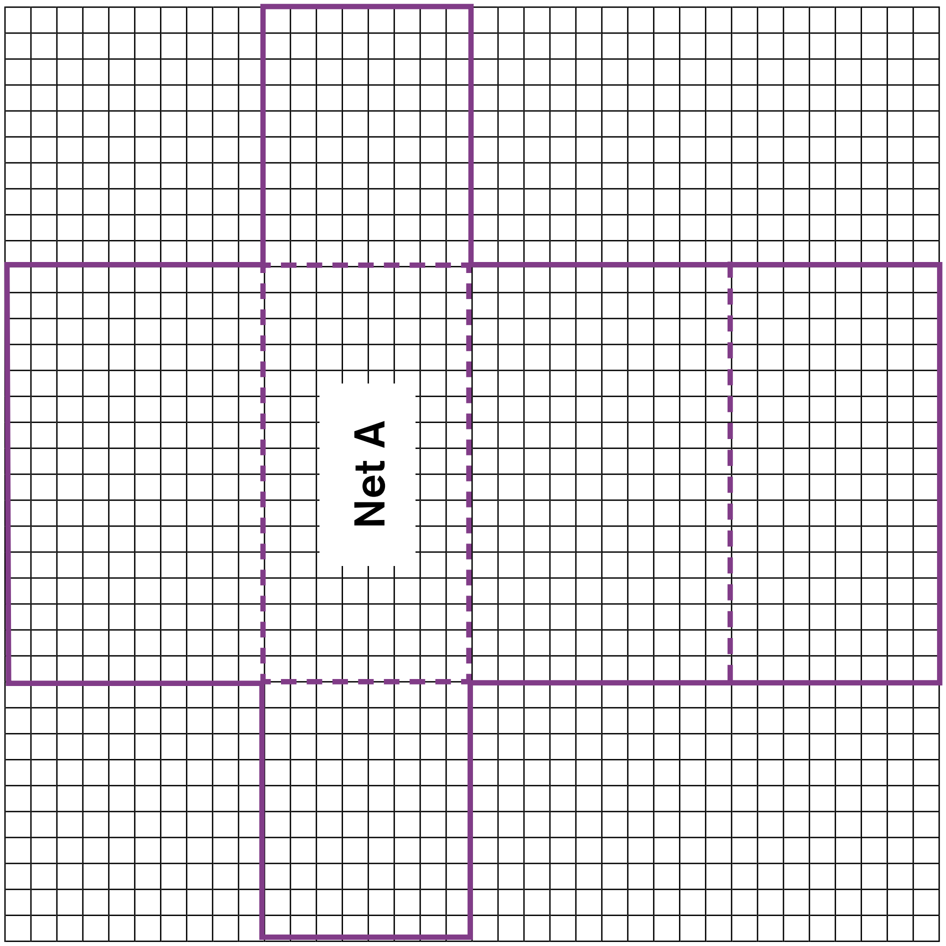
Volume: ________________

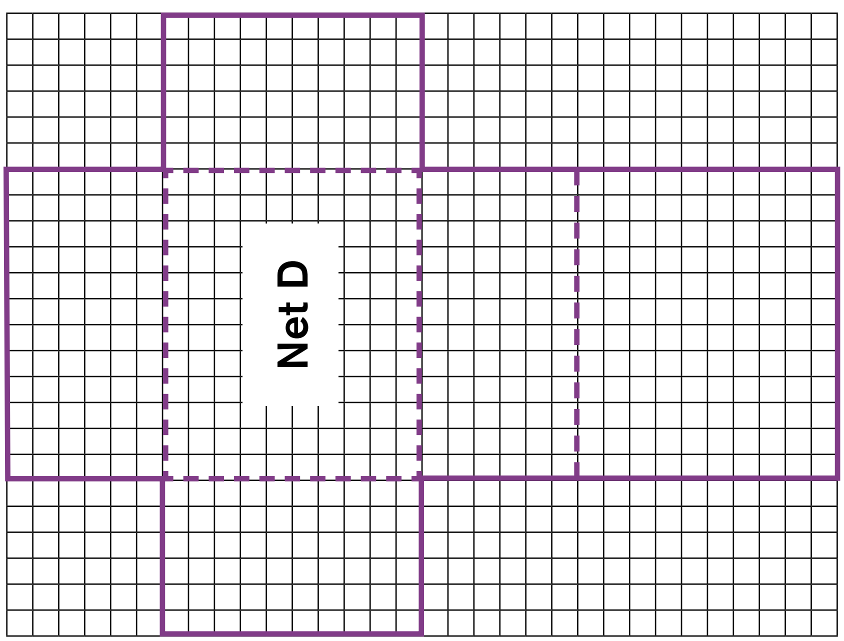
Surface Area: ________________

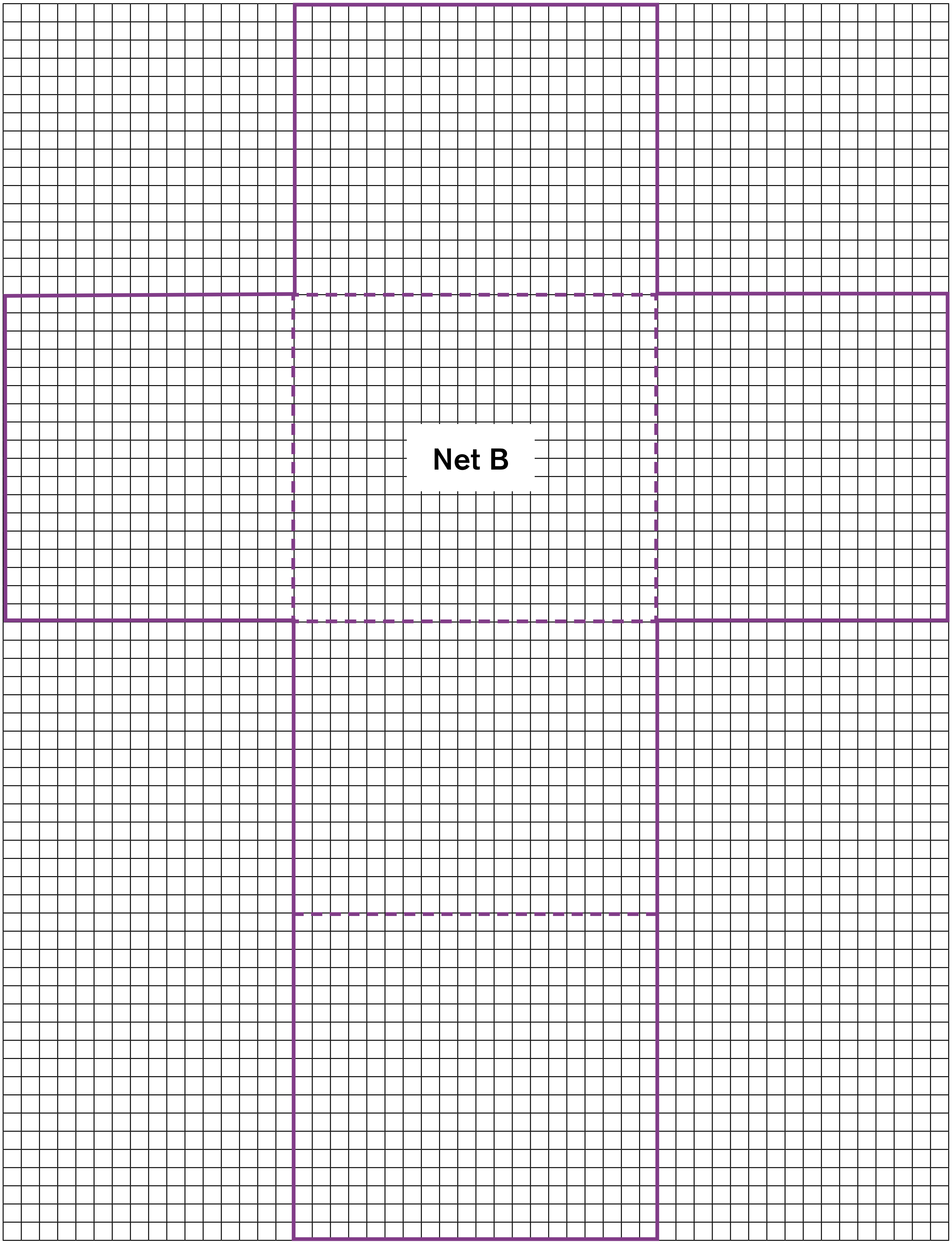
BOX D

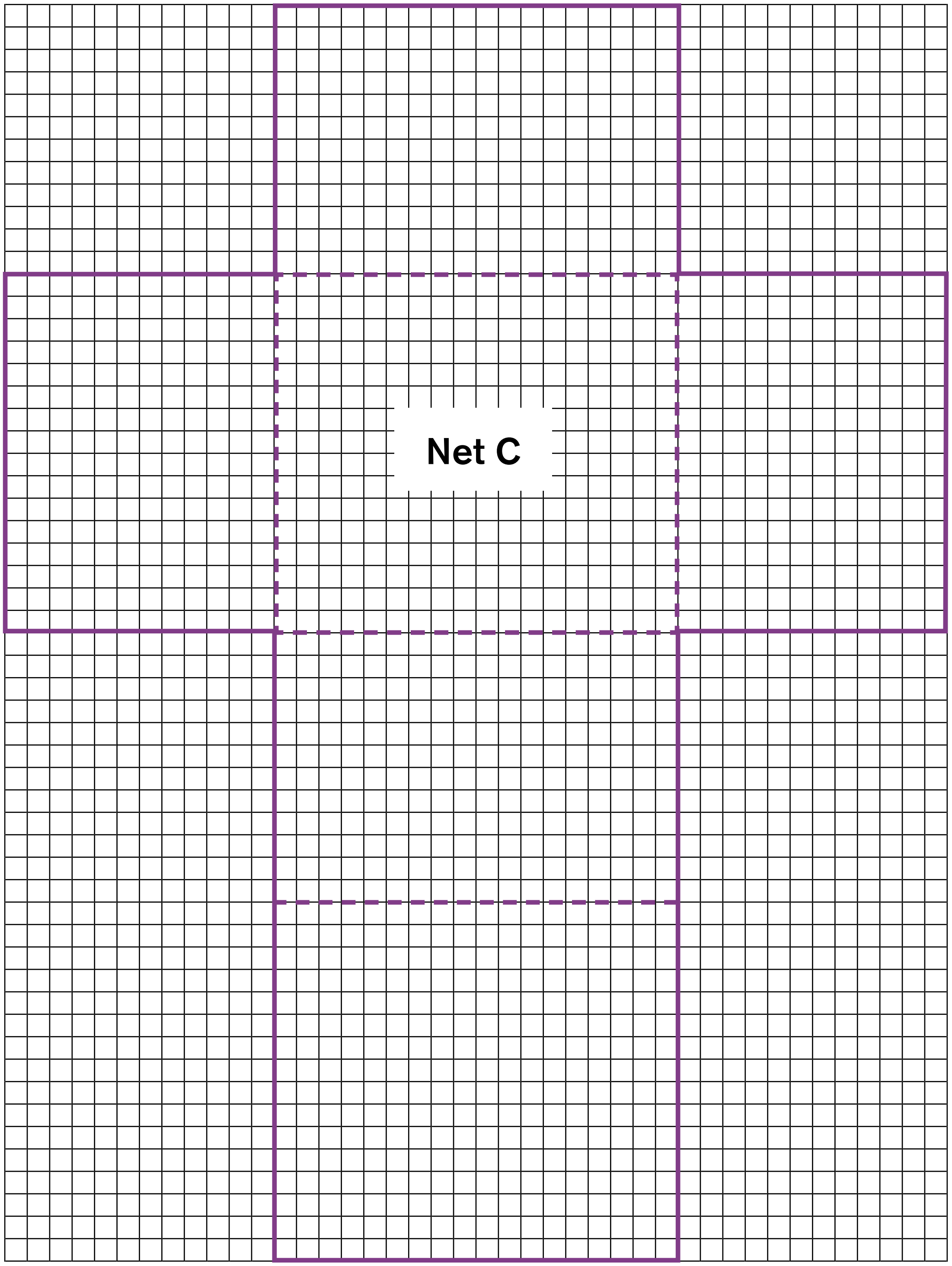
Volume: ________________

Surface Area: ________________


Net A


Net D


Net B

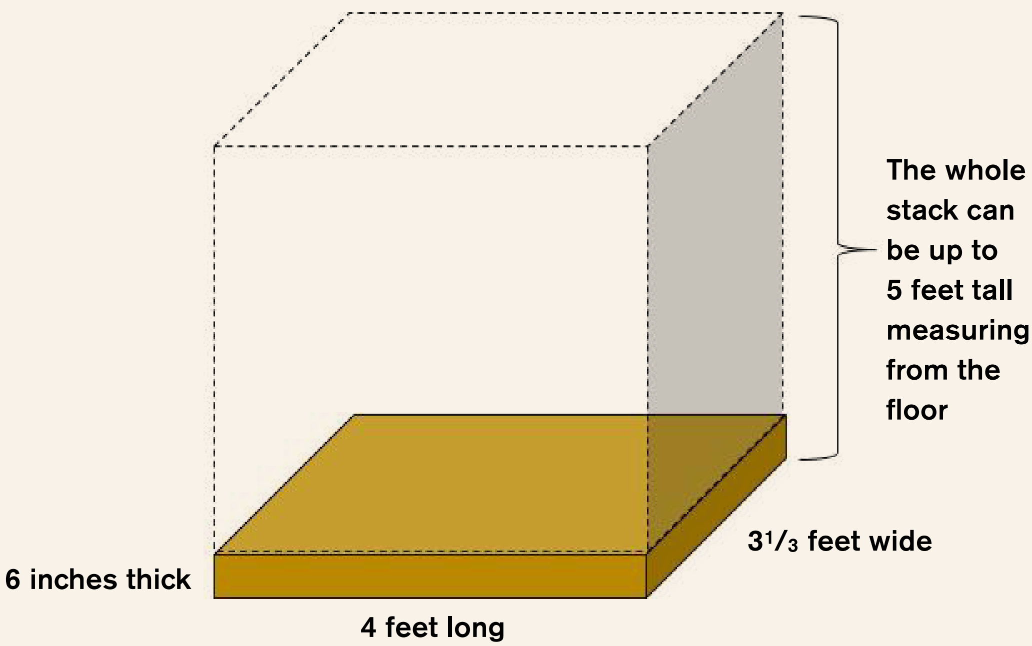

Net C

Packing Boxes on a Pallet

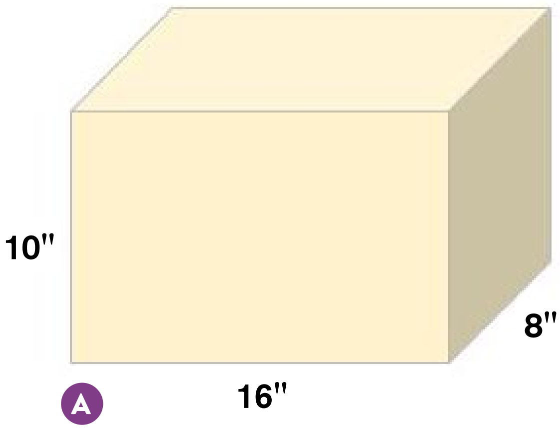
Before transporting products from a factory to a grocery store, companies stack boxes on wooden pallets, secure the boxes with shrink wrap and load them onto trucks for delivery. It's important that the boxes are stacked carefully so that the contents are stable during transportation. Companies try to fit as much on each pallet as they can so they can minimize the cost of transportation. When the pallets arrive at the store, the boxes are unloaded and signed for by a worker who takes inventory. The contents are put on shelves for sale or stored for future shelving.

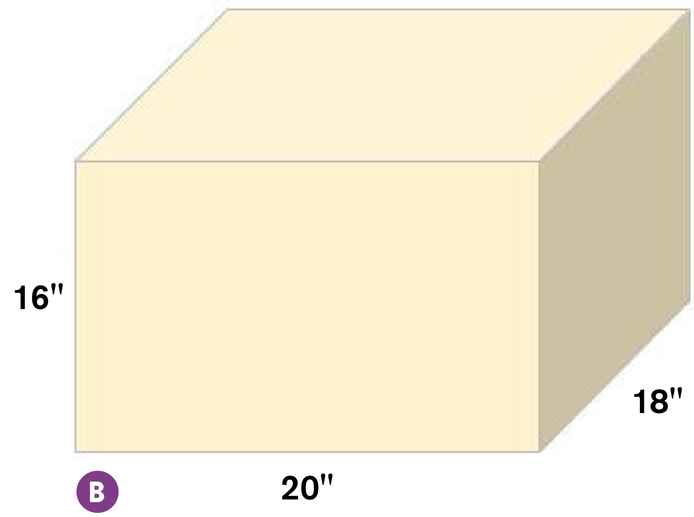
INFORMATION FOR STACKING BOXES:

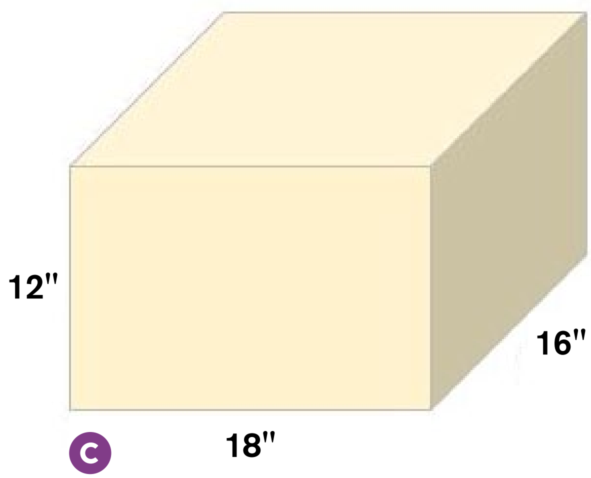
- It is recommended that pallets never be stacked higher than 5 feet high, to reduce the danger of boxes falling and injuring someone.
- Standard wooden pallets measure 4 feet by $3^1/_3$ feet and are 6 inches thick.
- Boxes should never overhang the edge of the pallet, since this can cause damage to the contents.

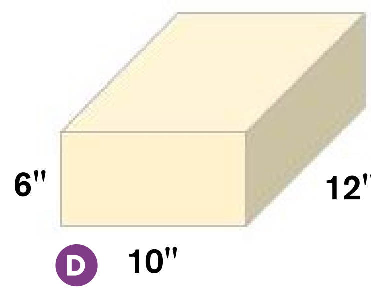


Cardboard boxes come in a variety of sizes.



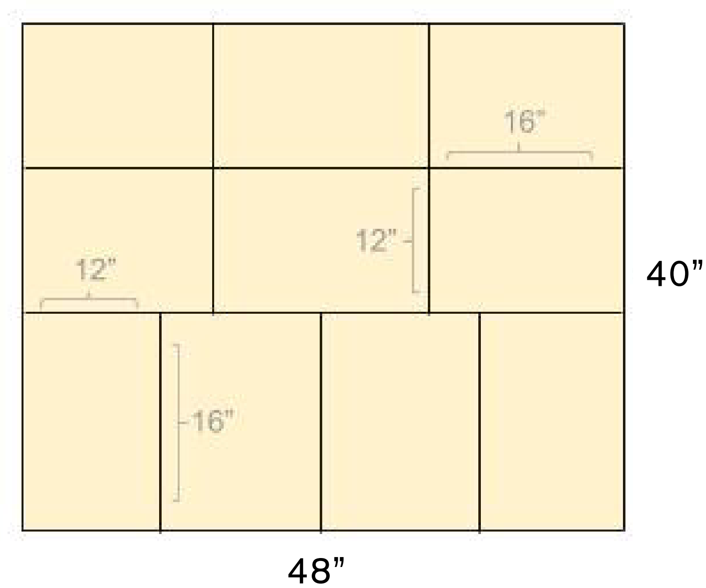




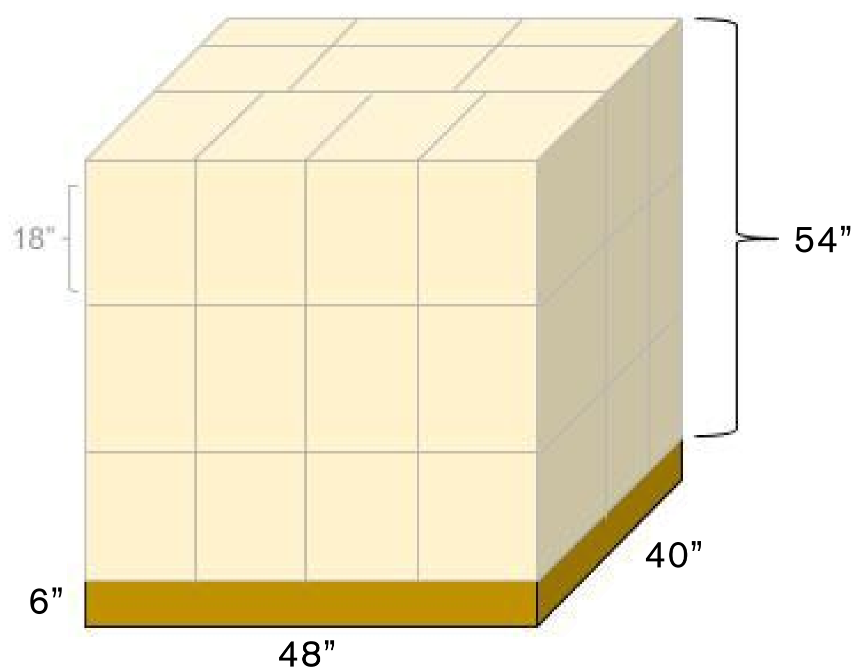


A Solution to Shipping Pallets of Boxes

View of pallet from above



View of pallet from the side



Converting Feet and Inches

Fill in the blanks.

Feet	Inches
1 ft	12 in
2 ft	
4 ft	
	96 in.
½ ft	
	3
¾ ft	
1 ½'	
	30"
1 $^{1}/_{3}$'	
	18
5' 2"	

Try some of your own:

Bonus:

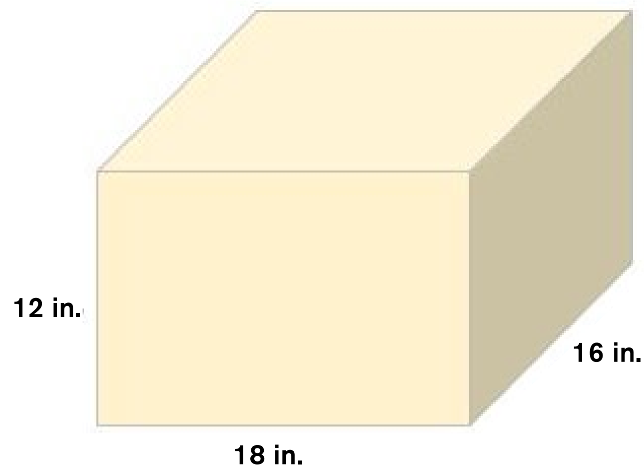
1 ft^2	__________ square inches

Shipping Pallets of Boxes

James has 300 boxes of apples that need to be shipped to New York City. He is using boxes that are 12 inches high, 18 inches long, 16 inches wide.

James will load the boxes onto standard pallets (40 by 48”).

How many pallets will James need in order to pack 300 boxes for shipment? *Please explain with words, numbers and/or drawings.*



INFORMATION FOR STACKING BOXES:

- It is recommended that pallets never be stacked higher than 5 feet high, to reduce the danger of boxes falling and injuring someone.
- Boxes should never overhang the edge of the pallet, since this can cause damage to the contents.

Grid paper