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Algebra

Algebra is the fundamental language of mathematics. It enables us to create a mathematical model of a situation, provides the mathematical structure necessary to use the model to solve problems, and links numerical and graphical representations of data. Algebra is the vehicle for condensing large amounts of data into efficient mathematical statements.

As we move into the twenty-first century, there is a great deal of concern about how much algebra today's students need to know to be successful in the workplace. While experts differ about the specifics, they agree that instruction that includes algebra and algebraic thinking is necessary for everyone: "It is essential for students to learn algebra as a style of thinking involving the formalization of patterns, functions, and generalizations, and as a set of competencies involving the representation of quantitative relationships" (Silver 1997, 206).

While the formal study of algebra usually occurs in grades 8 through 10, research has shown that elementary students can think about arithmetic in ways that provide a foundation for learning algebra. In fact, introducing basic forms of algebraic thinking into instruction in the elementary grades has been shown to enhance students' learning of arithmetic (Carpenter et al. 2003). Researchers suggest that there are three processes that when applied to a mathematical task support the development of students' algebraic reasoning: generalizing, formalizing, and justifying.

Generalizing is the process of developing a general mathematical statement about the structure, properties, or relationships that underlie a mathematical idea.

Formalizing is the process of representing the mathematical generalizations in some sort of formal way. Students progress in their ability to represent ideas formally—from talking about them in everyday language; to increasingly using mathematical terms; to representing generalizations using pictures, drawings, and graphs; to recording their insights using symbols. For example, students might notice that when you multiply zero and any number, the product is zero. They generalize their observation to all numbers and talk and write about what they notice. Eventually students progress to formally representing this property using variables: $a \times 0 = 0$ where a is any number. Likewise, young students first notice that the number pattern 3, 8, 13, 18, 23, 28, 33, . . . , is growing by 5s. They represent the increase in the pattern in more and more sophisticated ways, perhaps noting that the pattern in the ones place alternates between 3s and 8s or by using the operation + 5.

The third process, *justifying*, goes hand in hand with conjecturing. When students conjecture, they propose a mathematical statement that they think might be true but has not yet been proven true. Justifying is the "process of developing mathematical arguments to explore and critique the validity of mathematical claims" (NCISLA 2003, 5). For many teachers and students, the first step is realizing that claims should and can be justified. Justification of generalizations or conjectures requires more than providing many examples. Students need to be able to explain why they know something will be true for all numbers.

A teacher's ability to help all students learn algebra depends in part on his or her awareness of the most important concepts and ideas: variables, symbols, structure, representation, patterns, graphing, expressions and equations, and rules and functions. Many of these concepts are introduced in the elementary grades, in particular the study of variables, patterns, rules and relationships, equality, and graphing. In the middle grades instruction is expanded and also focuses on representation, expressions, equations, and functions. Teachers who ask questions that assist students in generalizing, formalizing, and justifying their statements about problems and situations are laying the foundation for understanding more complex mathematics, including algebra.

1. The Concept of Variable

Historically, algebra has progressed through three major stages, each defined by the concept of variable prevalent during that period. Algebra in its earliest stage did not include symbols but rather used ordinary language to describe solution processes for certain types of problems. The second stage (circa A.D. 250–1600) included using symbols to represent unknown specific quantities, the goal being to solve problems for these unknowns. In the third stage (1600–present), symbols have been used to express general relationships, to prove rules, and to describe functions.

Francoise Viète (1540–1603) was the first to use letters in formal mathematics notation. Shortly thereafter, René Descartes used letters in a more systematic way: a, b, c for constants and x, y, z for unknowns. When Descartes went to have his manuscript *La Geometrie* published in 1637, the printer, Jan Maire, of Leyden, Holland, began running out of some of the letters in the type set. He asked Descartes whether it mattered which letters represented the unknowns. Descartes replied that the specific letter was unimportant, as long as the unknown was represented by x, y, or z. Having plenty of the letter x on hand, the printer used it to represent the unknowns, thus contributing to the formulation of the algebraic dictum, "Solve for x."

As the concept of variable developed historically, the ways in which letters were used expanded. Experts categorize variables in different ways, but any particular use of a variable is determined by the mathematical context. In elementary and middle school, variables are primarily used to represent specific unknown values in equations, sets of numbers in inequalities (e.g., $x < 10$), property and pattern generalizers ($a + b = b + a$), formulas ($A = l \times w$), and varying quantities in functions ($y = 2x + 1$). The most common uses of variables are as specific unknowns, as generalizers, and as varying quantities.

Activity

Considering Variables

Objective: identify examples of the different uses of variables.

In the following equations and inequalities, variables are used in different ways. How is the variable being used in the examples below? What is the value of the variable in each example?

1. $2 \times \square = 15 - \square$
2. $y \leq 32$
3. $2x + 3 = y$
4. $(a \bullet b) \bullet c = a \bullet (b \bullet c)$
5. $49 = s^2$
6. $A = lw$

Things to Think About

One important use of a variable is as a particular but unknown number. For example, in $x + 6 = 10$ and $3m = 18$, the value of x is 4 and the value of m is 6. There is a specific number in each case that makes the equation true. This is the most common use of variable in the elementary grades. In the early grades, the unknown value is often represented using a shape such as a square or triangle: $\square - 8 = 10$. Algebraic equations that include one or more variables are sometimes referred to as *open sentences*. Open sentences are neither true nor false until values are substituted for the variables. For example, if we replace $\square$ with 18, the open sentence $\square - 8 = 10$ is true but if we replace $\square$ with any other number, the equation is false. Our goal when solving for unknowns is to find values that make equations or open sentences true!

Examples 1 and 5 use variables as unknowns: in 1, $\square = 5$; in 5, $s = 7$ and $^-7$. Were you surprised that example 5 has two solutions? In the equation $49 = s^2$, both $s = 7$ and $s = {}^-7$ make the equation true. Even though there is more than one solution, the variable still represents specific unknowns—just two in this case.

How is the variable used in example 2? In this inequality, y is equal to a set of numbers that starts at 32 and decreases infinitely. The idea that a set of numbers is the solution to a mathematical statement can be difficult for students if they have only had experiences with variables as specific unknowns.

The equation in example 3 represents a function in which the value of one variable depends on the value of the other variable. One letter takes on a set of values and has a systematic relationship with the other letter ($y = 2x + 3$)—the value of y depends on the value of x. When the value of one variable changes in relation to the value of another variable, we refer to this as joint variation. In the chart below are values for x and y. The y values were calculated by replacing the x in the equation $y = 2x + 3$ with different x values. Notice how as the value of x increases, the value of y increases.

x	y
$^-4$	$^-5$
0	3
2	7
9	21

The set of values for x is called the *domain* and the set of values for y is called the *range*. This interpretation of variable as a varying quantity is essential to understanding the relationship of patterns to functions.

In example 4, variables are used to illustrate a general property, in this case the associative property of multiplication. The variables do not represent an unknown or a varying quantity that is related to another variable. This is a generalization in which the letters convey a relationship that is always true about the multiplication of real numbers.

In the last example, the formula $A = lw$ provides information on the area of a rectangle and describes the relationship between three quantities: the *area* of the rectangle, the *length* of the rectangle, and the *width* of the rectangle. Letters in a formula always represent varying numbers and delineate a real-world relationship. For example, the equation $F = ma$ represents the relationship between force, mass, and acceleration in physics. Thus, we could classify example 6 with example 3: variables as quantities in joint variation.

It is important for students to realize that there are a number of interpretations of the term *variable*. If they understand variables only as representations of specific unknowns, it is not surprising that they have difficulty interpreting inequalities in which variables represent sets of numbers. On the other hand, sometimes students take the term *variable* literally and assume that a variable is something that differs or varies. We compound the problem by not referring to a variable as an unknown when it is appropriate to do so. Other difficulties arise when students overgeneralize; students first come to realize that the symbol or letter in a problem (e.g., □ above) represents the same number throughout an equation no matter how many times it appears. Yet this leads them to mistakenly assume that when there are different variables in an equation (e.g., $a + b = 12$), they cannot be replaced by the same value (in this case, 6). Furthermore, they mistakenly conclude that in the equations $m + 3 = 7$ and $y + 3 = 7$, the m and y must be different quantities because different letters are used. ▲

A common misconception involving variables is that students interpret them as labels. If g represents the number of girls in a dorm room, $3g$ is interpreted by students as 3 girls instead of 3 times the number of girls in the dorm room. In addition, the use of particular letters in conversions ($3f = 1yd$) and formulas (l is used for length) sometimes contributes to students' mistakenly thinking of a variable as a label. For example, in the expression, $2t + 4$ where t represents the number of tiles, students often interpret $2t$ to mean 2 tiles rather than 2 times the number of tiles.

Teachers should consider how to transform problem situations to support students' understanding of variable, especially as a generalizer of patterns and properties. One way is to vary the numerical values of givens in problems—and to examine the patterns that emerge. For example, if a problem states that CDs cost \$12 and a student only has \$5, teachers can ask how the amount needed will change if CDs cost \$13, \$14, \$15, or \$16. Students can be asked to write number sentences for the amount needed to buy a CD for each CD value and observe what changes and what stays the same each time. Eventually they can be asked to generalize the situation ($N - 5$ where N equals the original cost of the CD).

Young children are capable of making conjectures about basic properties of number operations. For example, students might recognize that multiplying any number by zero results in a product of zero. They can be asked to express their conjecture using words, numerical examples, and eventually with an open sentence that includes a variable (e.g., $a \times 0 = 0$). After students are comfortable with their conjecture and generalization, they can be challenged to justify it. Students often suggest many more specific examples; if this occurs, teachers must encourage students to show that their generalizations are true for all numbers, not just some (Carpenter et al. 2003).

2. Symbols

Mathematical symbols provide us with an efficient way to convey information without using words. When people talk about the language of mathematics, they are often referring to the symbols and shorthand notations that we use to do mathematics. These symbols must be learned and then repeatedly interpreted within problems and procedures. There are many different kinds of mathematical symbols: numerals and variables (often called literal symbols) such as $^-5$, 36, x, □, and V; operational symbols such as + and ÷; relational symbols such as = and >; and geometric symbols such as ∠ and ⊥, to name just a few. Shorthand notations such as LCM (least common multiple) and P(A) (probability of A) are also common. In Section 1 we discussed variables; in this section we describe operational and relational symbols.

What complicates the interpretation of symbols is that some symbols have more than one meaning. Parentheses are used to differentiate the operation of multiplication from a numerical value—2(3) versus 23—and to indicate a grouping that is to be performed first following the order of operations—$(3 + 9)^2$. In addition, some situations can be expressed using more than one symbol. For example, multiplication is indicated using the "times" sign—2×3—by a raised dot—$2 \bullet 3$—and by placing two symbols next to each other without spaces—2(3), lw, and $2x$. Take a minute and think of an operation that can be shown using a variety of symbols. Division can be expressed using ÷, /, and $\overline{)}$. As adults we forget to explore explicitly with students the fact that there are multiple ways to represent an operation symbolically. Can you think of any symbols that are interpreted differently depending on their use? The fraction bar sign ($\frac{3}{5}$) can indicate a division, a fraction, or a ratio. Certain letters represent types of quantities in specific formulas but can represent any number in other equations. For example, the C in $C = 2r$ stands for the length of the circumference, the C in $F = 32 + (\frac{9}{5})C$ represents degrees in Centigrade, but the c in $6c = 18$ is an unknown value (3) and is unrelated to a particular context. While adults have internalized these subtle and not so subtle differences, students often are unaware of multiple meanings. Thus, it is extremely important that we engage students in discussions about the meaning of symbols.

It is important for students to think carefully about operational symbols. Operation symbols describe an action on one or two numbers or symbols. When we see these operation symbols, we are keyed to "do something." Students encounter operational symbols for addition and subtraction (+ and −) in their first year or two of schooling. These operations are *binary operations*: the operation is performed on two numbers. Multiplication and division (× and ÷) are also binary operations—we can multiply or divide only two numbers at a time. Other operations, such as roots, absolute values, and powers ($\sqrt{25}$, $|^-4|$, and 8^3), are unitary operations. Unitary operations are conducted on one number at a time. What kind of operations are $(2 \div 6)^2$ or $|9 + {^-7}|$? These expressions combine operations—the division $(2 \div 6)$

is a binary operation that equals $\frac{1}{3}$ and the exponent (power of 2) is a unitary operation: $(\frac{1}{3})^2 = \frac{1}{9}$. The parenthesis, a grouping symbol, lets us know which operation to perform first. Likewise, the addition $9 + {}^{-}7$ is a binary operation which is completed first. The absolute value is then taken on the result, 2.

Another important group of symbols are classified as *relational symbols*. Relational or relation symbols establish a relationship between two numbers, two number sentences, or two variable expressions. Some common relational symbols are $=$, $\neq$, $\leq$, $\geq$, $<$, and $>$. The relationship is either true ($10 = 10$) or false ($5 > 8$), though in the case of open sentences ($9 + 3 = \square + 4$) we always try to find a value to make the sentence true. What can be confusing is that on either side of the relational sign there may be operations that must be completed in order to evaluate whether or not the relationship is true or false. Examine the following and consider whether the relationships that are being established are true or false.

$6 \neq 14 \div 2$	True, 6 does not equal $14 \div 2$, or 7
$18 \geq 19$	False, 18 is not greater than or equal to 19
$5 + 7 = 2 \times 6$	True, $5 + 7$ equals 12 and 2×6 equals 12, $12 = 12$
$\square < \square + 3$	True, any number is less than that same number plus 3

Students need to examine many different number sentences such as the first three examples above and evaluate whether they are true or false. Teachers can then ask students to change the sentences, making true ones false and false ones true. Or students might be asked to group number sentences and explain why they placed different sentences together. Asking students to discuss the relationship between the quantities on each side of a relational symbol will help them interpret these symbols correctly. Open sentences in which one or more variables are represented are especially problematic for students. Research has shown that in open sentences like $8 + 4 = \square + 5$, students in grades 1 through 6 overwhelmingly think $\square$ equals either 12 or 17 (Falkner, Levi, and Carpenter 1999). They add the first two or all three numbers and do not interpret the equal sign as a symbol that establishes a relationship between the quantities on either side of it. This important idea is explored in more detail in the next section.

3. Equality

Another fundamental idea of algebra is *equality*. Equality is indicated by the equal sign and can be modeled by thinking of a level balance scale. Why is equality important for students to understand? First, the idea that two mathematical expressions can have the same value is at the heart of developing number sense. For example, we want students to realize that there are many ways to represent the same product ($9 \times 4 = 2 \times 3 \times 6$). We want students to use what they know about the composition of numbers to help them remember number facts and form equivalent statements ($7 + 6 = 6 + 6 + 1$ and $7 + 6 = 7 + 7 - 1$). Understanding these number sentences and the relationships expressed by them is linked to the correct interpretation of the equal symbol.

The second reason for understanding the concept of equality is that research has shown that lack of this understanding is one of the major stumbling blocks for students when solving algebraic equations. To solve $3x + 2 = 14$, we add $^{-}2$ to both sides of the equal sign, thus maintaining balance or equality ($3x + 2 + {^{-}2} = 14 + {^{-}2}$, or $3x = 12$). The next step in the solution process involves multiplying both $3x$ and 12 by $\frac{1}{3}$, again because performing the same operation on equivalent expressions means they will remain equivalent ($(\frac{1}{3})3x = 12(\frac{1}{3})$, or $x = 4$). If students do not understand the idea of equality of expressions, they may perform computations on one rather than on both sides of an equal sign.

Yet the concept of equality is not easy for students and many do not correctly interpret the equal symbol. For example, often students think that the equal sign is a symbol that tells them to do something (such as subtract or multiply) rather than a symbol that represents equal values or balance. Students see an equation such as $6 + 4 = \square + 6$ and assume 10 is the answer because they have completed the addition on the left: to them the equal sign means "fill in the answer." Other children cannot make sense of $7 = 9 - 2$ because the operation symbol is to the right of the equal sign rather than the left. When students make these types of mistakes, we need to ask them to explain their thinking and to share with us what the equal sign means to them. However, telling students that the equal sign is, by convention, the symbol that lets us know that quantities are equal is not sufficient to clear up their misconceptions. We need to include activities and discussions in our instruction that focus on understanding equivalent values. Exploring whether number sentences are true or false is one activity that helps build understanding with number sense.

Another activity that supports understanding equality is to use balance scales. The balance scale is a visual model for the equality relationship. Most students intuitively understand that a balanced scale remains balanced if equal amounts are added to or subtracted from both sides of the scale. Balance scale problems can be used to investigate equivalence and to prepare students informally for symbolic representation and more abstract solution techniques.

Activity

2

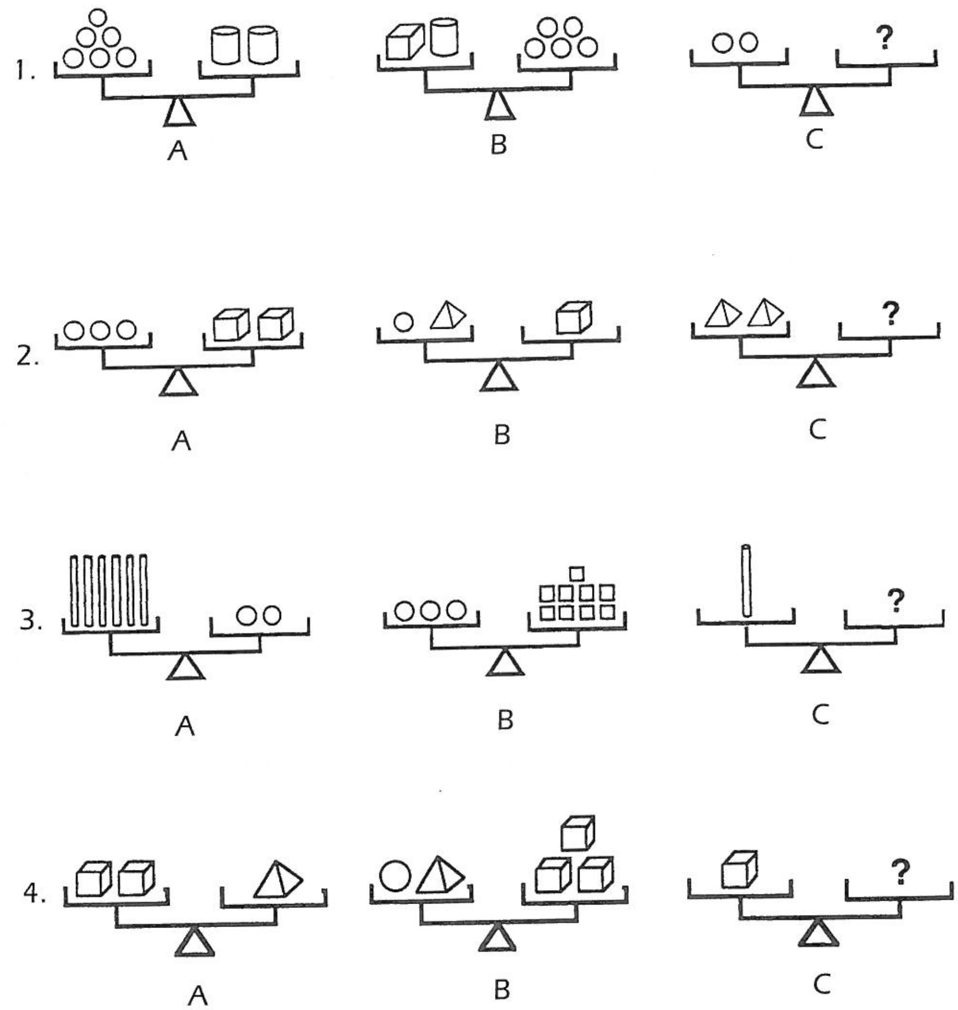
Balance Scales

Objective: explore the concept of equality using a balance scale.

Solve the balance scale problems on page 197. On each of the balance scales, assume that the same shapes represent the same weights. In each problem, use the information from the balanced scales A and B to figure out what's needed to balance scale C.

Things to Think About

If you have never examined problems like these, there are a number of general principles you have to consider. First, a level scale implies that the quantities on one pan are equivalent in weight to the quantities on the other pan. Second, we can modify both sides of the balance scale and maintain equality using either additive reasoning—removing (or adding) the same amount from (to) both sides of the scale—or multiplicative reasoning—multiplying or dividing both sides by the same factor. (If two cylinders balance six spheres, then one cylinder is equivalent in weight to three spheres). Third, we can replace objects of equal weight. These



principles are not easy for students to understand, especially when presented in the form of equations and inequalities. However, they are more accessible and engaging when students encounter them in the form of balance scales.

Let's examine balance problem 1 in detail. Scale A shows that six spheres are equivalent to two cylinders. Dividing the number of spheres and the number of cylinders in half results in three spheres balancing one cylinder. We can use this equivalence relationship to adjust scale B. Let's remove the cylinder from the left side and three spheres from the right side of the balance (since they are equivalent). The remaining cube balances two spheres: two spheres are equivalent in weight to one cube. So a cube added to the right side of scale C will balance it.

Do we always need to start with scale A? Absolutely not! In problem 2, let's focus first on the relationship shown on scale B. Using this relationship to substitute specific shapes on scale A, we find that the two triangular pyramids on scale C equal one sphere. In problem 3, we can also start with scale B, establishing that one sphere weighs the same as three cubes. Using substitution on scale A, we can replace the two spheres with six cubes. Six sticks are equivalent to six cubes, so the single stick on scale C is equal to one cube.

We solve balance scale problems by reasoning about equal relationships. Another way to do this is to link these actions and relationships to algebraic symbols. The relationship depicted in scale A of balance problem 4, in which two cubes weigh the same as one pyramid, can be indicated as $c + c = p$. Scale B shows this same relationship, but in addition a cube has been placed on the left pan and a sphere on the right pan and the scale still balances. This implies that a cube weighs the same amount as a sphere. The weights on scale B can be indicated symbolically as $s + p = c + c + c$. When comparing the two equations, we can reverse the order of the second equation and then substitute information from the first equation into the second equation:

scale A		$c + c = p$
scale B		$c + c + c = s + p$
	so	$c + p = s + p$
		$c = s$

This tells us that a cube weighs the same as a sphere. ▲

4. Patterns and Rules

Mathematicians repeatedly make the point that one of the primary activities in mathematics is to describe patterns: patterns in nature, patterns we invent, even patterns within other patterns. By examining a wide range of patterns, we notice regularity, variety, and the ways topics interconnect. We also see that certain patterns occur again and again. There are many types of patterns: repeating patterns, sequence patterns, and special patterns like Fibonacci numbers. Some patterns can be represented using rules or functions. Other patterns can be represented both numerically and geometrically and help us link arithmetic and geometry.

One type of pattern that is introduced in the early grades is a repeating pattern. Repeating patterns have a part, sometimes called the *core*, that repeats over and over. For example, in the pattern ♥♣♥♣♥♣♥♣♥♣ . . . , the core of ♥♣ repeats. Students must identify the core of a pattern in order to continue it. Repeating patterns can be presented orally (e.g., the refrains of many songs repeat) or by using motions such as clapping and turning. Repeating patterns can also be represented using numbers, pictures, and objects. Young children need to identify, describe, extend, and create a wide variety of repeating patterns.

Activity

Investigating Repeating Patterns

Objective: learn about the structure of repeating patterns.

Examine the pattern 1-2-3-4-1-2-3-4- . . . and make a list of three questions you might ask students regarding the pattern. Imagine extending this pattern indefinitely. What will the 19th number be? the 81st number? If you examine the first 42 numbers in the pattern, how many of the numbers will be 2s?

Things to Think About

What questions did you think of to pose to students? *Identify the core of the pattern? Continue the pattern? Fill in a missing value?* Questions like these are very appropriate for young children who are still learning to make sense of patterns. However, students also benefit from questions that force them to generalize relationships and use repeating patterns in more complex ways.

Determining the 19th number in the pattern takes two steps. First you have to notice that the core of the pattern involves four numbers, 1-2-3-4. Then you have to replicate the core a number of times to come close to 19 numbers. Did you make four copies of the core, using 16 numbers, and then count on 3 more numbers to reach the 19th number? Or did you make five replicas of the core, giving you the 20th number, and then count back 1 number? Either way a 3 is the 19th number in this pattern. Using this same form of reasoning, the 81st number is a 1.

How many 2s will be needed to write the first 42 numbers in this pattern? You will need ten sets of the core, which means you will use ten 2s. But since you also need an additional 1-2 to have a total of 42 numbers, the total number of 2s needed is eleven. ▲

Nonrepeating patterns are more difficult to understand. Students must not only determine what comes next in the pattern but also begin to generalize. Numerical relationships become more and more important. A particular type of pattern in which numbers, objects, letters, or geometric figures are in an ordered arrangement is called a *sequence* or, informally, a *growing pattern*. Growing patterns are found in instructional materials in grades 3, 4, and 5 and in middle school. The numbers, objects, letters, or geometric figures that make up the sequence are called the *terms*, *steps*, or *stages* of the sequence. Sequences are classified according to the methods used to determine subsequent terms. There are *arithmetic* sequences and *geometric* sequences.

In an arithmetic sequence, the same number, called the *common difference*, is added to each previous number to obtain the next number. Some numerical examples are:

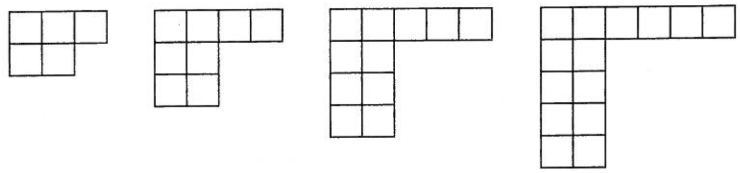
5, 6, 7, 8, 9, 10, . . .   2, 4, 6, 8, 10, . . .   7, 10, 13, 16, 19, 22, . . .

In the first sequence the common difference is 1: the first term is 5, the second term is 6, the third term is 7, and so on. The second sequence has a common difference of 2, since the number 2 is added to each previous term. In the third sequence, the common difference is 3: term number one is 7, term number two is 10, and term number three is 13.

Sometimes arithmetic sequences decrease rather than increase, because the number that is added each time is a negative number. In the examples below a $(^-5)$ and a $(^-\frac{1}{2})$ are being added to the previous term. Extend each pattern.

$15, 10, 5, 0, {}^-5, {}^-10, \ldots$ $10, 9\frac{1}{2}, 9, 8\frac{1}{2}, 8, 7\frac{1}{2}, \ldots$

Sequences can also be represented using geometric designs or figures. How many squares will be in the 15th figure of the following pattern?



Students sometimes find it easier to notice the numeric pattern, or common difference, from one term to the next term when they record the steps in a table.

STEP	1	2	3	4	5	6	. . .	15
NO. OF SQUARES	5	8	11	14	?	?	. . .	?

Did the geometric design or the table help you notice that the pattern is successively increasing by 3? When a procedure is applied over and over again to a number or geometric figure to produce a sequence of numbers or figures, we say that the procedure is *recursive*. Each stage of a recursive procedure builds on the previous stage. Recursive procedures are sometimes referred to as *iterative* procedures because the same rule is repeated again and again. In the geometric pattern above we repeatedly add 3 squares to create each successive design. Students often identify recursive rules (the instructions for producing each term or step of a recursive sequence from the previous term) and then use the rule to find each subsequent term (in this case, we can determine each successive term by adding 3 to the previous term). This recursive strategy works to find the number of squares in a small number of terms like the 15th step (5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47) but is not efficient or reasonable if we have to find the number of squares in the 100th term. We would have to list the first 99 terms in order to find the 100th term!

The form of an arithmetic sequence can be expressed using variables. This enables us to generalize patterns and relationships. Many teachers prefer to use symbols such as triangles and squares before introducing letters as variables. Students can be first challenged to articulate the recursive rule (e.g., *this sequence is increasing by adding* 6 *to each term*) and then asked to represent the pattern more explicitly using words, symbols, and a variable. The mathematical purpose is to help students formalize the similarities and regularities they have observed.

How might we generalize the arithmetic sequences 2, 4, 6, 8, 10, . . . and 7, 14, 21, 28, 35, . . . ? In each of these sequences the common difference also happens to be the first term. Using a symbol ($\triangle$, for example) or a letter (x, for example) to represent both the difference and first term, we can express the sequence like this:

$$\triangle, \triangle + \triangle, \triangle + \triangle + \triangle, \triangle + \triangle + \triangle + \triangle, \ldots$$
$$x, x + x, x + x + x, x + x + x + x, \ldots$$

In the sequence above, the variables—$\triangle$s or xs—are repeatedly added to form the terms of the sequence. Another way to represent this is by using multiplication; first we have one $\triangle$, then two $\triangle$s, then three $\triangle$s, and so on, or $1x$, $2x$, $3x$, $4x$, and so on.

Variables can represent different things depending on how they are used. In this case, the variable ($\triangle$ or x) represents a set of numbers. Substituting different numbers for the common difference creates different sequences:

$1x, 2x, 3x, 4x, \ldots$	$1x, 2x, 3x, 4x, \ldots$
$1(2), 2(2), 3(2), 4(2), \ldots$	$1(7), 2(7), 3(7), 4(7), \ldots$
$2, 4, 6, 8, \ldots$	$7, 14, 21, 28, \ldots$

How might we determine the 32nd number in the first sequence (2, 4, 6, 8, . . .) other than using the recursive rule of + 2 and listing the first 31 terms? Is there another pattern that can be used to link the term of the sequence to the actual number in the sequence? Yes, an explicit rule where we multiply the term number and the common difference can be used. The 32nd number in the sequence will be 32×2, or 64, since 2 is the common difference. The 32nd term in the sequence starting with 7 above is 32×7, or 224.

Not all arithmetic sequences begin with the common difference; most, such as 1, 3, 5, 7, 9, . . . , start with a different first number (in this case, 1) and repeatedly add the common difference (in this case, + 2). We can represent this sequence using symbols by letting $\square$ represent the starting term and letting $\triangle$ represent the common difference (the number added to each term). We can also use letters. Let a equal the starting term and let x equal the common difference. In this case, there are two variables (shapes or letters) in our generalized statement.

$\square,$	$\triangle + \square,$	$(\triangle + \square) + \triangle,$	$(\triangle + \triangle + \square) + \triangle,$	$(\triangle + \triangle + \triangle + \square) + \triangle, \ldots$
$\square,$	$\triangle + \square,$	$2\triangle + \square,$	$3\triangle + \square,$	$4\triangle + \square, \ldots$
$a,$	$x + a,$	$2x + a,$	$3x + a,$	$4x + a, \ldots$

Let's substitute numbers into this generalized pattern. If $a = 1$ and $x = 2$ then:

$$a, x + a, 2x + a, 3x + a, 4x + a, \ldots$$
$$1, 2 + 1, 2(2) + 1, 3(2) + 1, 4(2) + 1, \ldots$$
$$1, 3, 5, 7, 9, \ldots$$

The generalization above works for all situations, whether the $\square$ represents zero, the common difference, or any other number. The use of variables to show the extension of an arithmetic sequences lays the foundation for students to be able to describe sequences using algebraic expressions or rules.

Activity

4

Exploring Arithmetic Sequences

Objective: practice using both recursive and explicit rules and generalized patterns to solve problems involving arithmetic sequences.

1. Make up an arithmetic sequence with a common difference of $^{-}6$. What would the 125th term be?
2. Make up an arithmetic sequence that starts with 3 and has a common difference of 5. Use the generalized pattern to find the 2,345th term in the sequence.

3. How many numbers (or terms) are there in the arithmetic sequence 4, 8, 12, 16, . . . , 124? What if you wanted to extend this sequence? Write a rule that you could use.
4. Examine the sequence 1, 5, 9, 13, 17, . . . , 65. What do you notice? How is this sequence similar to the one in Question 3? different?

Things to Think About

If a sequence has a common difference of $^{-}6$, each subsequent term of the sequence is smaller than the previous term (since we are adding a $^{-}6$ each time). Using our recursive rule and starting with $^{-}6$, some terms in this sequence would be: $^{-}6$, $^{-}12$, $^{-}18$, $^{-}24$, $^{-}30$, ... To find the 125th term, we need an explicit rule linking the term number and the sequence number—namely $125 \times (^{-}6)$. The 125th term in this sequence is $^{-}750$.

The arithmetic sequence for Question 2 is 3, 8, 13, 18, 23, 28, 33, 38, and so on. In each case, a common difference of 5 is being added. Did you notice the pattern in the ones digits of the numbers in this sequence? The ones digit is either a 3 or an 8. The 2,345th term can be found by multiplying 2,344 by 5 and then adding 3, because 3 was the starting point of the sequence ($2{,}344(5) + 3 = 11{,}723$); the 2,345th term is 11,723 (which ends in a 3!).

Did you notice that the sequence in Question 3 has a common difference of 4? Each value in the sequence is 4 more, or 4 times the number of the term ($4 = 4 \cdot 1$, $8 = 4 \cdot 2$, $12 = 4 \cdot 3$, . . .). To find how many numbers or terms are in the sequence, divide 124 by 4. There are 31 numbers in the sequence.

So far we have written rules in which the variable, x, represents the common difference. But what if, as in Question 3, you know the common difference? Then you might want to let the variable represent the position of the term in the sequence. Since the common difference in Question 3 is 4, we can represent any term as $4x$. Put another way, our explicit rule for any number in the sequence is $4x$ where x represents the term number. The idea that variables can represent anything (the common difference or the term or both) is what makes algebra so useful and also so confusing.

The sequence in Question 4 is very similar to the sequence in Question 3. It also has a common difference of 4, but the sequence begins with 1 rather than 4. All of the terms are 3 less than the corresponding terms in the Question 3 sequence:

1, 5, 9, 13, 17, . . .

4, 8, 12, 16, 20, . . .

Can we use the rule above in which x represents the position of the term in the sequence as the basis for a similar rule for the sequence in Question 4? Since the numbers in this sequence are each 3 less than the numbers in the $4x$ sequence, we can show this algebraically as $4x - 3$. Substitute various term numbers for x to show that this rule will generate the sequence. ▲

Exploring arithmetic sequences forms the basis for later work with functions, in which two variables are related in such a way that one depends on or is affected by the other. In arithmetic sequences, many things can be represented as a variable: the common difference, the starting number, the position in the sequence, and the actual value of each term. Some of these variables depend on one or more of the other variables. Investigating these relationships informally helps prepare students to explore these dependent relationships formally in middle and high school.

The other type of sequence is the geometric sequence. In a geometric sequence, instead of the same number being *added* to each term to obtain the subsequent term, the same number is *multiplied* by each term to obtain the subsequent term. Rather than having a common difference, as arithmetic sequences do, geometric sequences have what is known as a *constant multiplier*—a number that when multiplied by the previous term produces the next number in the sequence. The sequences below highlight how a common difference and a constant multiplier affect the numbers in a sequence. In the arithmetic sequence 1, 3, 5, 7, 9, . . . the common difference is 2. Now let's start a geometric sequence with 1 and let the constant multiplier be 2. Since we are multiplying each number by 2, the outcome is quite different: 1, 2, 4, 8, 16, ... Determine the common difference and the constant multiplier in the following sequences.

Arithmetic Sequences	*Geometric Sequences*
1, 3, 5, 7, 9, . . .	1, 2, 4, 8, 16, . . .
5, 6, 7, 8, 9, . . .	5, 5, 5, 5, 5, . . .
2, 4, 6, 8, 10, . . .	2, 4, 8, 16, 32, . . .
7, 10, 13, 16, 19, . . .	7, 21, 63, 189, 567, . . .
7, 14, 21, 28, 35, . . .	7, 49, 343, 2401, 16807, . . .

The form of a geometric sequence also can be expressed using variables—symbols or letters. Let the $\square$ or the letter a represent the starting term and $\triangle$ or the letter x represent the constant multiplier, the number that is multiplied by the previous term.

$$\square, \square \bullet \triangle, (\square \bullet \triangle) \bullet \triangle, (\square \bullet \triangle \bullet \triangle) \bullet \triangle, \ldots$$
$$a, a \bullet x, (a \bullet x) \bullet x, (a \bullet x \bullet x) \bullet x, \ldots$$

Since we are multiplying by the same number each time, we can represent the pattern using exponents. Exponents are a shortcut for repeated multiplication. In 8^3, the number 8 is called the *base* and the number 3 is called the *exponent*, or power. Furthermore, the symbols 8^3 together are referred to as an *exponent* or an *exponential*. Exponents represent a product (in the case of 8^3, 256). An important exponent definition is that any number, a (where $a \neq 0$), to the zero power is equivalent to 1 (e.g., $a^0 = 1$, $5^0 = 1$, $35^0 = 1$, $0.2^0 = 1$). Below are the two generalized statements for the sequences using exponents:

$$(\square \bullet \triangle^0), (\square \bullet \triangle^1), (\square \bullet \triangle^2), (\square \bullet \triangle^3), (\square \bullet \triangle^4), \ldots$$
$$(a \bullet x^0), (a \bullet x^1), (a \bullet x^2), (a \bullet x^3), (a \bullet x^4), \ldots$$

Some numerical examples are:

3, 6, 12, 24, 48, . . .	$(3 \bullet 2^0), (3 \bullet 2^1), (3 \bullet 2^2), (3 \bullet 2^3), (3 \bullet 2^4), \ldots$
1, 5, 25, 125, 625, . . .	$(1 \bullet 5^0), (1 \bullet 5^1), (1 \bullet 5^2), (1 \bullet 5^3), (1 \bullet 5^4), \ldots$
40, 10, 2.5, 0.625, . . .	$(40 \bullet 0.25^0), (40 \bullet 0.25^1), (40 \bullet 0.25^2), (40 \bullet 0.25^3), \ldots$

In descending geometric sequences it can appear that the terms are being repeatedly divided. In fact, the terms are being multiplied by a fraction ($\frac{1}{4}$). This has the same effect as dividing by the reciprocal of the fraction (in the example above, the reciprocal of $\frac{1}{4}$ is 4).

The terms in a geometric sequence increase or decrease much more quickly than the terms in an arithmetic sequence. This type of growth is known as *exponential* *growth*—and the decrease is known as *exponential decay*. Exponential growth and decay occur when the change between terms is the result of multiplication, not addition. Students informally learn about exponential growth and decay when solving problems or when working with geometric sequences. They study these concepts formally in middle and high school.

Activity 5

Choosing Between Salary Options

Objective: compare the growth of an arithmetic sequence and a geometric sequence.

Tom agrees to care for his neighbors' cat while they go on a two-week vacation and has to choose one of two pay options. Payment plan A consists of \$2 on the first day, \$4 on the second day, \$6 on the third day, \$8 on the fourth day, and so on for 14 days. Payment plan B is also by the day: 2¢ on the first day, 4¢ on the second day, 8¢ on the third day, 16¢ on the fourth day, 32¢ on the fifth day, and so on. Calculate how much money would be made each day according to the different plans. Which plan would you choose? Why?

Things to Think About

Were you surprised by the difference in the sizes of the salaries on the 14th day? Plan A increased slowly and steadily while plan B increased very slowly at first but then grew rapidly. The salary according to plan A is based on an arithmetic sequence in which the common difference is 2. Each day the salary increases by two more dollars; therefore, on day 14 you make \$28 dollars. Generalizing using plan A, on day x you receive $2x$ dollars. The total amount of money for the 14 days is found by adding the amounts earned on day 1 through day 14—\$210.00!

Plan B is based on a geometric sequence in which the constant multiplier is 2. Each day the salary from the day before is multiplied by 2. This salary plan pays less per day until day 12. However, the last three days pay so well that the total amount of money earned for the 14 days is \$327.66! A comparison of the salary plans is shown below:

DAY	1	2	3	4	5	6	7
Plan A	\$2	\$4	\$6	\$8	\$10	\$12	\$14
Plan B	\$0.02	\$0.04	\$0.08	\$0.16	\$0.32	\$0.64	\$1.28

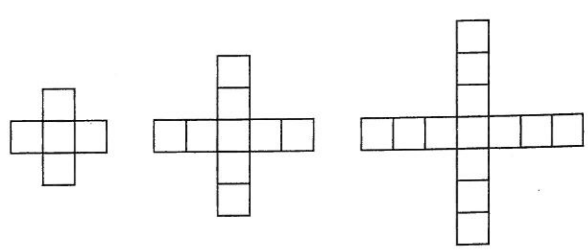
DAY	8	9	10	11	12	13	14
Plan A	\$16	\$18	\$20	\$22	\$24	\$26	\$28
Plan B	\$2.56	\$5.12	\$10.24	\$20.48	\$40.96	\$81.92	\$163.84

The doubling pattern in plan B may be easier to recognize when the dollar and cent signs are removed: 2, 4, 8, 16, 32, 64, 128, ... This sequence could be represented using exponents: 2^1, 2^2, 2^3, 2^4, and so on. In fact, the powers of 2, as this sequence is often called, is one that we want older students to recognize and link to exponents automatically. The salary on the 14th day is equivalent to

2^{14}, or 16,384¢. The exponent, 14, is the same number as the day (or term). The explicit rule can be generalized as 2^n where n is the day or term.

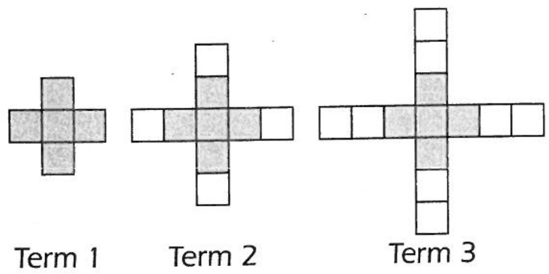
There are a number of children's stories that deal with exponential growth. *Anno's Magic Seeds*, by Mitsumasa Anno, *The King's Chessboard*, by David Birch, *A Grain of Rice*, by Helena Clare Pittman, and *The Rajah's Rice*, by David Barry, are just a few. These stories provide a context in which elementary and middle school students can explore geometric growth and work with patterns that involve exponents. Students' understanding of different types of growth is expanded when comparing arithmetic and geometric sequences. In addition, students enjoy learning how to represent products using exponents. ▲

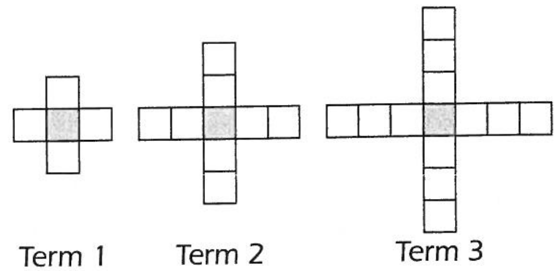
When students explore patterns and generalize relationships among numbers, they are developing informal understanding of one of the most important topics in high school and college algebra—functions. A *function* is a relationship in which two sets are linked by a rule that pairs each element of the first set with exactly one element of the second set. We use functions every day without realizing it. The relationship between the cost of an item and the amount of sales tax is a function, the automatic calculations performed by computer spreadsheets are based on defined relationships between and/or among data fields, and our car's gasoline mileage is a function that depends on the speed of the car and the efficiency of the engine. When we were talking earlier about rules for arithmetic and geometric sequences, we were generating function rules, also referred to as explicit rules. The first set of numbers are the term or stage numbers (e.g., 1st, 2nd, 3rd, 4th, . . .) and the second set of numbers are the values of each term or, in other words, the numbers in the sequence (e.g., 10, 20, 30, 40, . . .). How do we assist students in the elementary and middle grades in generalizing explicit rules? One method is to start with patterns that are created using concrete objects such as pattern blocks, toothpicks, or square tiles. Students can be encourage to extend the pattern using objects or drawings and talk and write about what they notice. For example, examine the pattern and draw the next two terms:



Term 1 Term 2 Term 3

How would you describe the change from one term to the next (the recursive pattern)? Some adults notice that the center five squares form a cross and the number of squares at the four ends of the cross increase by one starting at term 2. Other people notice a center square in each cross and that the lengths of the four legs of each cross increase by one as the term increases. Different ways of seeing the pattern are highlighted on the following page.





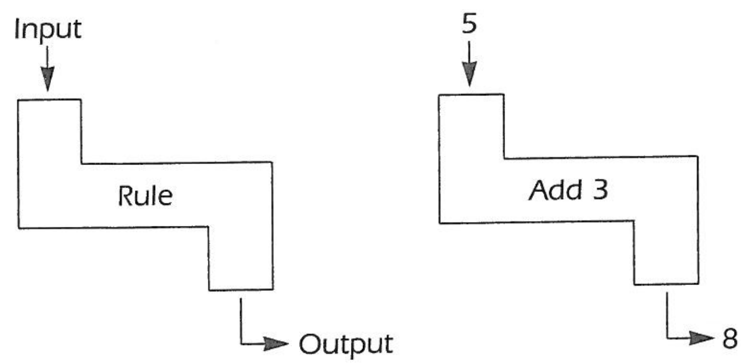
Asking students to describe in words what they observe and to look for parts of the design that stay the same and parts that change often assists students in generalizing.

However, in order to have students generalize the explicit or function rule, we must move beyond simply extending the pattern and instead help them to make sense of a rule between the term number and the pattern or sequence number. An expanded t-chart is especially useful. In an expanded t-chart the columns indicate both the constant amount and the change from term to term. Students can fill in their t-charts with the observed data and then predict the pattern for several larger terms or stages. We also want them to use these data to derive a rule for the pattern.

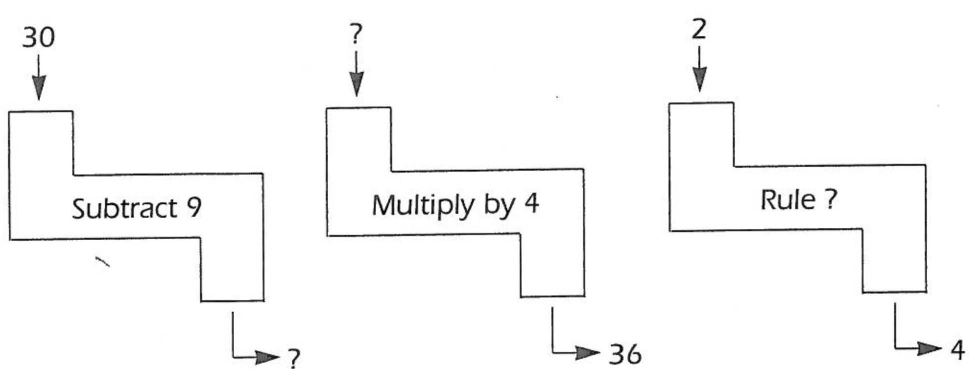
TERM NUMBER	CONSTANT + CHANGE IN SQUARES	PATTERN IN CHANGE	TOTAL NUMBER OF SQUARES
1	1 + 4	4 × 1	5
2	1 + 8	4 × 2	9
3	1 + 12	4 × 3	13
4	1 + 16	4 × 4	17
5			
15			
25	1 + 100	4 × 25	101
n		$4 \times n$	$1 + 4n$

From the t-chart, we can see that for term number n, the number of squares can be represented with the explicit rule of $1 + 4n$. Students need many opportunities to extend patterns and to analyze constant and changing values in order to generalize and write explicit rules.

Another way for students to learn about functions and to practice generating explicit rules is by using function machines. A function machine is an imaginary device that links sets of numbers: an element (the input) is put into the machine and acted on according to a rule, and another, related element (the output) is produced:



When any two of these three components (input, rule, output) are known, the third component can be analyzed (and often determined). In the examples that follow, finding the output in the first function machine (21) is very straightforward. Finding the missing input, as in the second function machine (it's 9), can be more complex, because we have to apply the inverse operation of the rule to the output. In the third example, however, since we are given only one input/output pair, we can't be certain what the rule is. That's because more than one rule can be applied (e.g., add 2, multiply by 2, or, perhaps, multiply by 3 and subtract 2).



Function machines are useful both for applying rules (forward and backward) and for generating rules for data sets. However, we also want older students to generalize these function rules using variables. One way to present these functional relationships is by using input/output tables and asking students to examine the input and output data, look for patterns and relationships, and write a rule for the *n*th input value.

Activity

6 Finding Function Rules

Objective: determine the explicit rule for each function machine.

Examine the following tables. Determine the relationship between the two sets of numbers, represent that relationship using a rule, and fill in the blanks. Consider whether the numbers are increasing or decreasing because of a common difference or a constant multiplier.

TABLE A

INPUT	OUTPUT
1	$^{-}2$
2	1
3	4
4	7
5	10
6	
. . .	. . .
9	
10	
	40
n	

TABLE B

INPUT	OUTPUT
1	3
2	9
3	27
4	81
5	243
6	
. . .	. . .
9	
10	
	531,441
n	

TABLE C

INPUT	OUTPUT
1	100
2	96
3	92
4	88
5	84
6	
. . .	. . .
9	
10	
	52
n	

Things to Think About

Numerous patterns jump out at us when we look at an input/output table. Looking at the two vertical columns in Table A, we see that the inputs are increasing by 1 each time and the outputs are increasing by 3. The repeated adding of 3 suggests that the relationship can be modeled by multiplying the input by 3. In order to determine a rule, however, we have to examine both the vertical and the horizontal patterns. What is consistent about how the input of 3 and the output of 4, the input of 4 and the output of 7, and the input of 5 and the output of 10 are related? The outputs are always greater than the inputs, but not by a constant amount. It sometimes helps to test a temporary rule suggested by the vertical outputs (multiplying by 3, in this case). Notice this is similar to what we did with the t-charts: identifying patterns in the change from term to term.

TABLE A

INPUT	OUTPUT
1	$^{-}2$
2	1
3	4
4	7
5	10
n	?

TEMPORARY RULE ($3n$)

INPUT	OUTPUT
1	$1 \times 3 = 3$
2	$2 \times 3 = 6$
3	$3 \times 3 = 9$
4	$4 \times 3 = 12$
5	$5 \times 3 = 15$
n	$3n$

Comparing the outputs in our temporary table to the corresponding outputs in Table A, we discover that they all differ by 5: $3 - (^{-}2) = 5$, $6 - 1 = 5$, $9 - 4 = 5$. Thus we can revise our temporary rule of $3n$ to $3n - 5$. Checking this rule against the inputs and outputs in Table A verifies that the rule holds. We then use it to fill in the blanks in Table A: 13, 22, 25, 15, $3n - 5$.

Did you notice that the outputs in Table B increase rapidly? This suggests that the rule might involve a constant multiplier. Looking at the vertical column of outputs we observe that each successive output is 3 times greater than the previous output, which suggests that the constant multiplier is 3. Writing each output as an exponent to the power of 3 (3^1, 3^2, 3^3, . . .), we can derive the function rule for Table B to be 3^n. The missing values in the table are 729, 19,683, 59,049, 12, and 3^n.

In Table C the inputs are increasing by 1 but the outputs are decreasing by 4. We can say we are either subtracting 4 each time or adding a ($^-4$) to each subsequent output. Since the common difference is ($^-4$), we can try out a temporary rule in which we multiply the input by $^-4$:

TABLE C

INPUT	OUTPUT
1	100
2	96
3	92
4	88
5	84
n	?

TEMPORARY RULE (^-4n)

INPUT	OUTPUT
1	$1 \times (^-4) = {^-4}$
2	$2 \times (^-4) = {^-8}$
3	$3 \times (^-4) = {^-12}$
4	$4 \times (^-4) = {^-16}$
5	$5 \times (^-4) = {^-20}$
n	^-4n

Looking at the vertical column, we see that we need to subtract $4n$ from a larger number. Try 100: $100 - 4n$. This is close, but doesn't work exactly: if $n = 1$, then $100 - 4 = 96$. Aha, but what if we change the starting number to 104: $104 - 4n$. Now the function rule works for all inputs. The missing values in Table C are 80, 68, 64, 13, and $104 - 4n$.

Another way to determine these function rules is to find the value at the 0 stage or term—namely to work backward through the tables to figure out the output when the input is 0. In Table C we saw that the common difference was ($^-4$). To move back up through the table, we perform the opposite operation or add 4 to each output. For example, term 5's output is 84, term 4's output is 88 ($84 + 4$), and term 3's output is 92 ($88 + 4$). Using this approach, term 0's output is 104. Why is the 0 term important? Recursive sequences are defined by a starting value and a rule regarding common differences or constant multipliers. We generate the sequence by applying the rule to the starting value, then applying it to the resulting value, and repeating this process. In order to write a function rule for a sequence we have to know both the rule (the common difference or the constant multiplier) and the starting value. Earlier in the chapter we explored sequences that began with the common difference or constant multiplier; the starting value of those sequences actually is zero but we don't bother to record it. The sequence 2, 4, 6, 8, 10, . . . , could be written as 0, 2, 4, 6, 8, and so forth. In sequences that do not begin with zero, the 0 term gives us this starting number. Table C starts at 104 and the rule is $104 + (^-4)n$, (or $104 - 4n$). Use this strategy to find the starting value of Table A. Did you work backward through the table subtracting 3 to get a value of $^-5$? The rule for Table A is $^-5 + 3n$, or $3n - 5$. By convention we usually write rules by listing the variable first, but sometimes beginning the rule with the starting value makes more sense.

In any function machine, the value of the output varies depending on the value of the input. For example, when we substitute different values for n (the input) we get different outputs for $104 - 4n$ (the output). Sometimes the input/output relationship is represented using the letters x and y. In the case of Table C, the functional relationship is $y = 104 - 4x$. The variables, x and y, take on another interpretation of the word *variable,* perhaps the most familiar one—variables as varying quantities. ▲

Patterns, variables, and functions are linked in a number of ways. Some patterns can be generalized using variables, and some patterns can also be represented as functional relationships. A variety of repeating, arithmetic, and geometric patterns can be used to explore these ideas. The ancient Greeks noticed that many numbers could be used to make interesting dot patterns. For example, nine dots can be arranged in rows and columns to make a square array, but eight dots cannot be arranged into the shape of a square, only into a rectangular array. Numbers of dots that can be arranged in geometric patterns are known as *polygonal numbers* or *figurate numbers*. Number patterns based on these geometric patterns are very common. Two patterns occur so frequently that they are known by the shapes they represent—square numbers and triangular numbers.

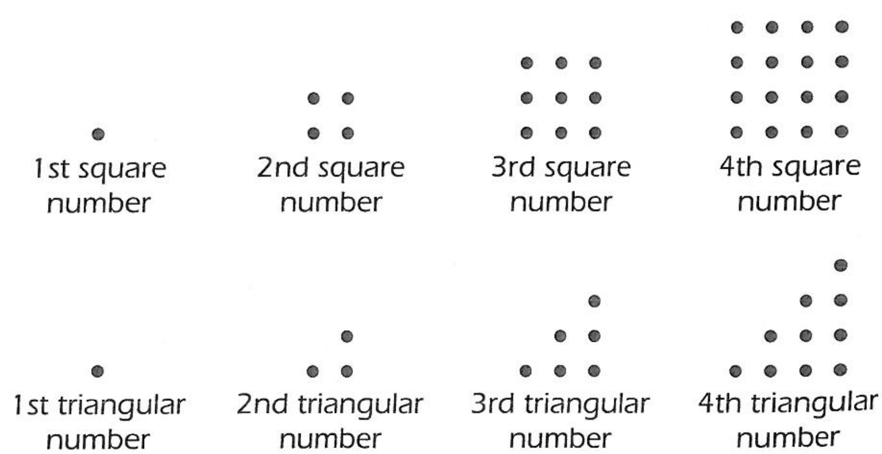
Activity 7

Exploring Square and Triangular Numbers

Objective: learn about the square and triangular number patterns.

The dot arrays below represent the first four square numbers and the first four triangular numbers. Complete the fifth and sixth sequences in each pattern and then answer the following questions.

1. Describe the 10th square number.
2. How many dots are in each of the square arrays that represent the square numbers? List the number of dots in a t-chart. What patterns do you see?
3. Describe the 10th triangular number.
4. How many dots are in each of the triangular arrays that represent the triangular numbers? List the number of dots in a t-chart. What patterns do you see?



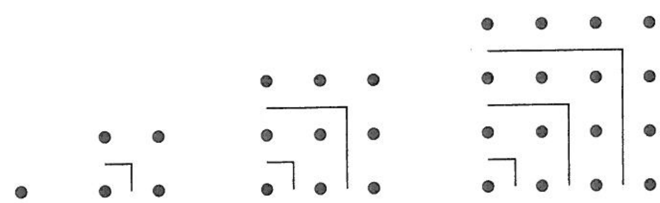
Things to Think About

Did you find that the 5th square number used 25 dots to make a 5-by-5 square and that the 6th square number used 36 dots to make a 6-by-6 square? Thus, the 10th square number is 100; 100 dots would be needed to make a 10-by-10 square. The square number pattern can be shown with dots in the form of square arrays but is more commonly represented by the number of dots: 1, 4, 9, 16, 25, 36, 49, and so on. These numbers are sometimes presented in exponent form: 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, etc. The explicit rule for generalizing the square number pattern is n^2, where n stands for the term (e.g., the 8th square number is 8^2, or 64). The second power (e.g., 6^2) of a number is often referred to as the number "squared" (6 squared equals 36), because that number of dots can be represented as a square.

Other interesting patterns are found in moving from one square number to the next. Notice that two consecutive square numbers differ by an odd number of dots.

1 (+ 3) 4 (+ 5) 9 (+ 7) 16 (+ 9) 25 (+ 11) 36

This can be shown visually in the construction of squares.



1   1 + 3 = 4   1 + 3 + 5 = 9   1 + 3 + 5 + 7 = 16

In order to answer Question 3 it helps to examine the fifth and sixth figures in the triangular numbers pattern. The fifth triangular number uses 15 dots to make a right triangle, with 5 dots on the bottom row, 4 dots on the next row, 3 dots above that, and then 2 dots and 1 dot respectively in the last two rows. The sixth triangular number uses 21 dots.

5th triangular number
15 dots

6th triangular number
21 dots

One way to make the 6th triangle in the triangular number pattern is to take the fifth triangle and simply add a bottom row of 6 more dots. This approach can be used to find the 10th triangular number. It is 55: 55 dots would be needed to make the 10th triangle. The triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, and so on.

What patterns did you notice in how these triangular numbers grow? The second triangle has 2 more dots than the first; the third triangle has 3 more dots than the second; the fourth triangle has 4 more dots than the third. These same patterns can be observed when looking at the numbers in the pattern.

TERM NUMBER	NUMBER OF DOTS	CHANGE
1	1	
		+2
2	3	
		+3
3	6	
		+4
4	10	
		+5
5	15	
⋮	⋮	

There is a rule for determining any triangular number:

$$\frac{n(n+1)}{2}$$

where n represents the term in the triangular number pattern (e.g., to find the third triangular number, substitute 3 into the rule

$$\frac{3(3+1)}{2} = \frac{12}{2}$$

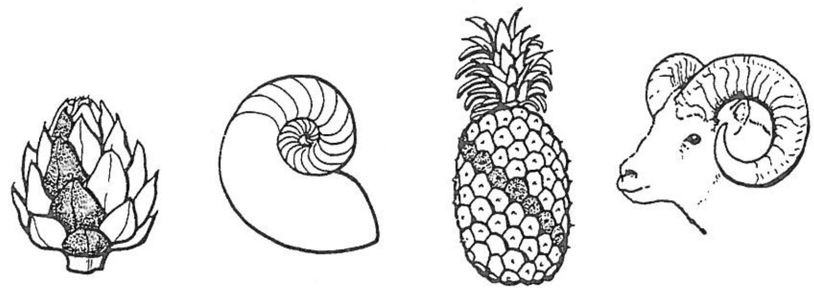
or 6; the third triangular number is 6). ▲

While many patterns can be categorized as repeating, arithmetic, geometric, or figurate, there are also special patterns that do not fit into these classification schemes. One of the most famous patterns is the Fibonacci sequence, which is made up of Fibonacci numbers. Fibonacci was the nickname of Leonardo de Pisa, an Italian mathematician (1175–1245); he is best known for the sequence of numbers that bears his name. The Fibonacci sequence of numbers begins with two numbers: 1, 1. Each new number is then found by adding the two preceding numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

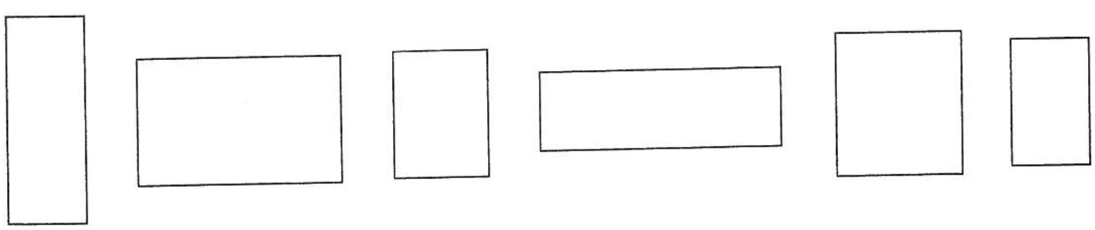
Mathematicians have identified many interesting relationships among the Fibonacci numbers. For example, the sum of the first three Fibonacci numbers ($1 + 1 + 2 = 4$) is one less than the fifth number (5). The sum of the first four Fibonacci numbers ($1 + 1 + 2 + 3 = 7$) is one less than the sixth number (8). Find the sum of the first five Fibonacci numbers. Did the relationship hold? Yes, the sum of the first five Fibonacci numbers is 12, which is one less than the seventh Fibonacci number.

The Fibonacci numbers describe a variety of phenomena in art, music, and nature. The numbers of spirals on most pinecones and pineapples are Fibonacci numbers. The arrangement of leaves or branches on the stems of many plants are Fibonacci numbers. On a piano, the number of white (8) keys and black (5) keys in each octave (13) are all Fibonacci numbers. The center of a sunflower has clockwise and counterclockwise spirals, and these spirals tend to be consecutive Fibonacci numbers. The lengths and widths of many rectangular objects such as index cards, windows, playing cards, and light-switch plates are consecutive Fibonacci numbers.



natural occurrences of Fibonacci numbers

Fibonacci ratios are comparisons of two Fibonacci numbers, usually adjacent numbers in the sequence. These ratios are often expressed as decimals by dividing one number in the ratio by the other. As the numbers in the Fibonacci sequence increase, the ratios tend to approximate 1.618 (8/5 = 1.6, 55/34 = 1.6176, 233/144 = 1.61805). This special ratio (1.618034 . . .) is an irrational number (see Chapter 1, page 5) that occurs in many other shapes and objects. It is known as ϕ (phi) by mathematicians, and was labeled the *golden ratio* by the ancient Greeks. (People also call it the *golden proportion*.) It has been known and used for thousands of years—it is believed that it was a factor in the construction of some of the pyramids in Egypt. Rectangles whose length-to-width ratios approximate the golden ratio are called golden rectangles. Psychologists have found that people prefer golden rectangles to other rectangles; thus common objects such as cereal boxes and picture frames tend to have dimensions with a ratio of around 1.6. Which of the rectangles below do you find to be most aesthetically pleasing? Two are golden rectangles; you can check by measuring the lengths and widths and calculating the ratios.



Activity 8

Fibonacci Numbers and You

Objective: investigate the occurrence of the golden ratio in the human body.

The human body is characterized by golden proportions, and these ratios have been used to draw figures accurately for centuries. Make the following measurements (use either inches or centimeters) and calculate the designated ratios. Can you find other golden ratios besides the ones mentioned?

- Your height compared with the distance from the floor to your navel.
- The distance from the floor to your navel compared with the distance from the floor to your kneecap.

- ▲ The length of your arm from the shoulder compared with the distance from your fingertips to your elbow.
- ▲ The distance from your chin to the center of your eyes compared with the distance from your chin to the tip of your nose.
- ▲ The length of your index finger compared with the distance from your index fingertip to the big knuckle.

Things to Think About

One way to represent these ratios is as decimals. If your height is 68 inches and the distance from the floor to your navel is 42 inches, the ratio is 68 to 42 or 1.619 to 1 (68 ÷ 42). This is very close to the golden ratio! You may instead have calculated a ratio of 1.7 or 1.5—not all individual proportions are exactly golden ratios. Some of us have long legs or arms compared with our overall heights. However, on average, the ratios will be close. Did you find other occurrences of the golden ratio? There are many other comparisons that produce the golden ratio, especially within the face and hands.

The golden ratio is more precisely defined as the cut in a line segment such that the ratio of the whole segment to the longer part is the same as the ratio of the longer part to the shorter part.

$$\frac{AC}{AB} = \frac{AB}{BC}$$

Thus, when we take body measurements, we are comparing a long section of the body to a shorter part. ▲

The relationship between Fibonacci ratios and the golden ratio is a curious phenomenon and has been the subject of study for generations. As a pattern, the Fibonacci sequence is an important one to know. Students benefit not only from identifying occurrences of the numbers in the pattern in nature but also from understanding the connection to the golden ratio. The golden ratio has been used in the design of many buildings (the Parthenon in Greece) and in the art of Leonardo da Vinci and Piet Mondrian.

5. Representation

Another important concept in algebra (and in mathematics in general) is *representation*—the display of mathematical relationships graphically, symbolically, pictorially, or verbally. Graphical representations include a variety of graphs—bar graphs, line graphs, histograms, line plots, and circle graphs (see Chapter 13 for additional information). Symbolic representations involve the use of symbols and include equations, formulas, and rules. Pictorial representations such as two- and three-dimensional drawings, maps, balance scales, and scale drawings are used in almost all areas of mathematics but especially geometry (see Chapters 10 and 11). Finally, relationships can also be expressed in words, either written or spoken. The situations you have been reasoning about in this chapter have been represented pictorially, symbolically, and verbally. Likewise, students need to create and match different representations in order to extend and deepen their understanding of relationships.

Activity 9

Representing Relationships in Problems

Objective: represent an algebraic relationship using a graph, table, and equation.

Generate the data needed to solve the following problem and organize the information into a table. Graph the data for each plan on a coordinate grid. Write equations to represent the relationships in each plan.

> *The Relax and Listen CD Club offers new members two plans from which to choose. Plan 1: Each CD costs $10.00. Plan 2: The first two CDs are free, and each additional CD costs $12.00. You are a big music fan and want to become a member of the Relax and Listen CD Club. Which plan would you choose? Explain.*

Things to Think About

Tables are a way of showing sets of data. In this problem you can set up a table that compares number of CDs and total cost for each plan. By listing the number of CDs in a systematic way, along with the corresponding cost, you can solve the problem and answer the question.

How many CDs should you include in the table? five? ten? twenty? In order to decide which plan is better for you, you have to find which quantity of CDs will cost the same under either plan. The table below shows that if you buy 12 CDs, you will spend the same amount under plan 1 and plan 2. This is often referred to as the "break even" point. If you plan to buy more than 12 CDs, plan 1 is the better choice. If you plan to buy fewer than 12 CDs, plan 2 is more economical.

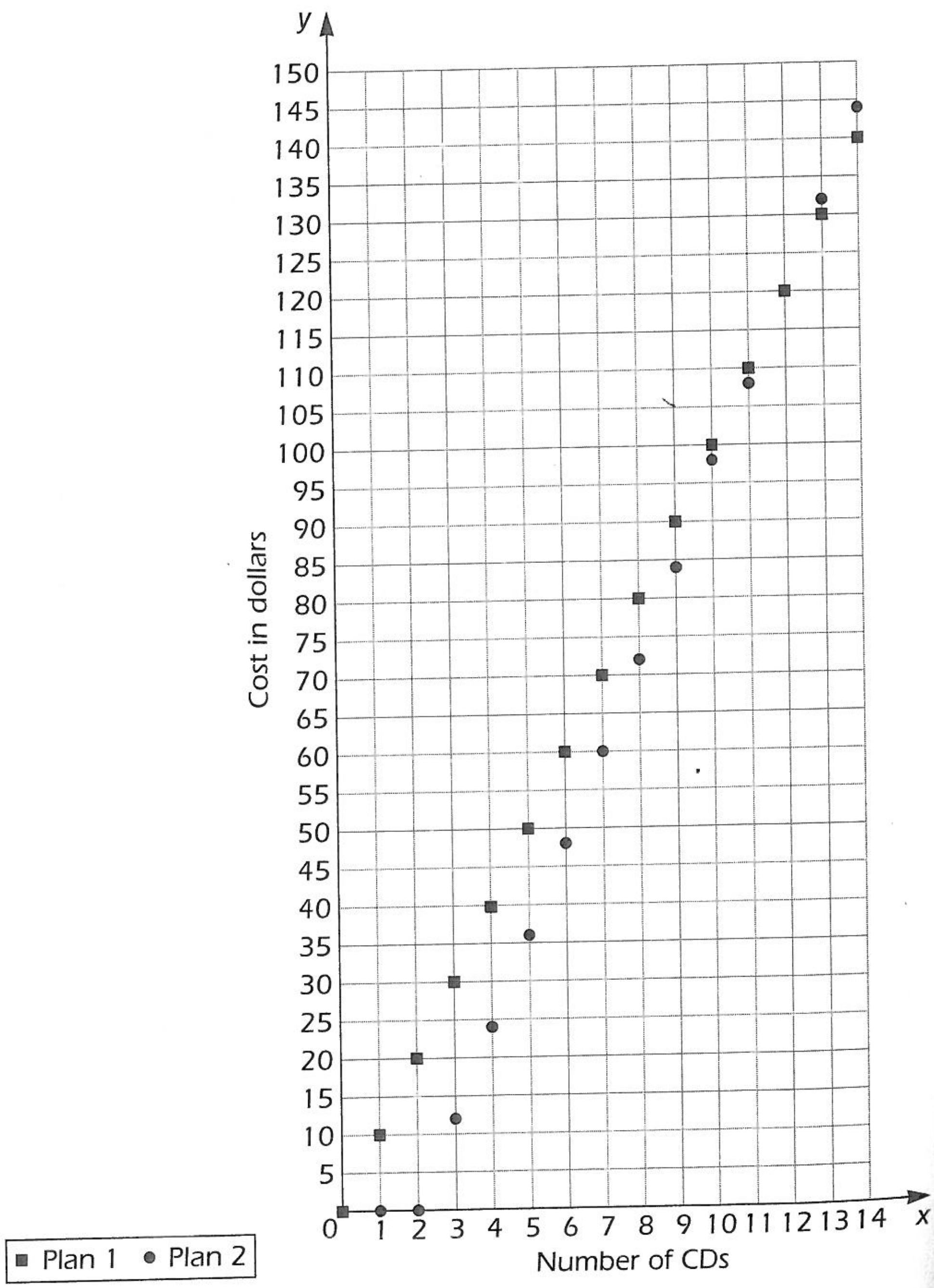
NUMBER OF CDS	TOTAL COST FOR PLAN 1	TOTAL COST FOR PLAN 2
1	$10	0
2	$20	0
3	$30	$12
4	$40	$24
...	...	...
11	$110	$108
12	$120	$120
13	$130	$132

What are the benefits of using tables of data to solve problems? Working with numbers in an organized way often helps students note patterns. Having calculated numerous costs for plans 1 and 2, students will be able to cite specific examples. You can then help them link the specific examples to general statements about the relationships in the plans. For example:

Student: Plan 1 went from $10, to $20, to $30, up to over $100.
Teacher: Can you describe what's happening in more general terms?
Student: Well, the cost increases $10 every time you buy another CD.

How can we generalize the relationship in each plan with a rule? Plan 1 increases steadily by \$10 with each CD; therefore, if n equals the number of CDs, the cost of plan 1 is $10n$ dollars. Plan 2 increases steadily by \$12; but two CDs are free; therefore, if n equals the number of CDs, the cost of plan 2 is $12(n - 2)$ dollars. To find the break-even point symbolically, we create a mathematical expression in which the two rules are set equal to each other: $10n = 12(n - 2)$.

Comparison of Two Music Club Plans



The solution to this equation is the number of CDs you can buy and spend the same amount under either plan:

$$10n = 12(n - 2)$$
$$10n = 12n - 24$$
$$^{-}2n = {}^{-}24$$
$$n = 12$$

Often rules or equations use x and y to represent the variables so that the equations can be more easily graphed. The equation for plan 1 is $10x = y$ and for plan 2 is $12(x - 2) = y$, where x is the number of CDs and y is the cost in dollars.

A graph will also show the relationship between the two plans. The title of the graph provides us with important information, as do the labels on each set of axes. (See page 216.)

The set of square points (plan 1) starts at the origin, or the (0,0) coordinate, and is plotted next at (1,10). This means that the cost of zero CDs is zero dollars and the cost of one CD is $10. The set of round points (plan 2) starts at the coordinate (2,0) and is plotted next at (3,12). This means that the cost of two CDs is zero dollars and the cost of 3 CDs is $12.

. Why are the points on the graph not connected into lines? The data in this problem are discrete or countable data (see page 277 in Chapter 12 for more information): whole number values for CDs make sense, fractional numbers of CDs (e.g., three and a half) do not make sense. We cannot connect the points, because the values between whole numbers of CDs have no meaning. However, we can plot these points on a line graph because we are examing two different variables—number of CDs and cost.

The intersection point on the graph represents the break-even point, the point at which the two plans are equal. Notice that the points for plan 2 form a steeper incline than those for plan 1. However, because the points don't start at the same coordinate, it appears that plan 1 is steeper at the start. The steepness, or the slope, of incline provides us with information about the rate of change (in this case, the changing cost) between points. The steeper the incline, the greater the amount of change (dollars here) between values. ▲

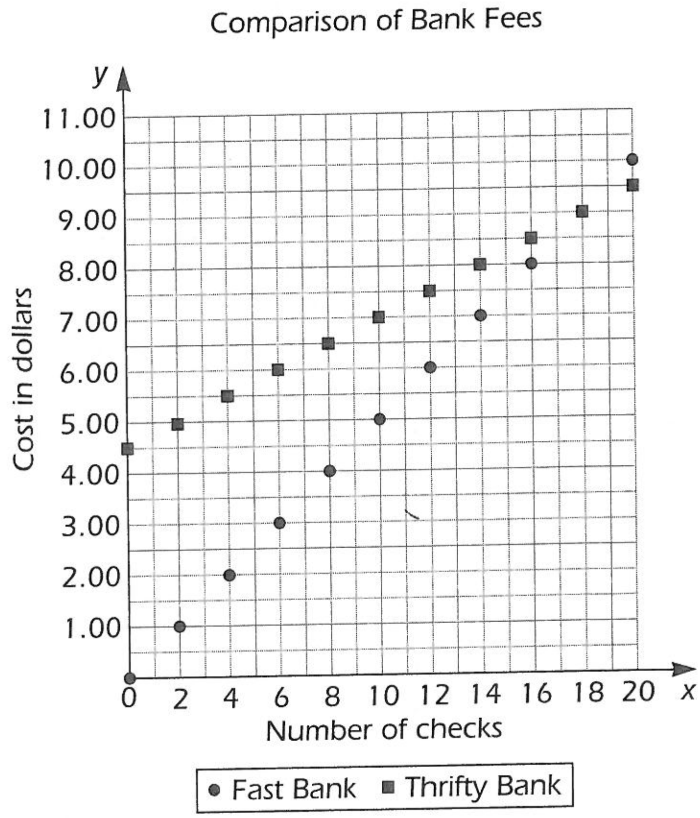
Even though the information and relationships in a math problem can be expressed using prose, tables, equations, and graphs, one representation is sometimes easier to use than another or provides us with different insights into a problem. How would you represent the information in the following problem in order to solve it?

> *For their checking accounts, the Thrifty Bank charges $4.50 per month and $0.25 per check. The Fast Bank charges a flat rate of $0.50 per check for their checking accounts, with no additional monthly fee. How many checks must a person write to make the Fast Bank checking account the more economical plan?*

The specific context of a problem can either contribute to or distract from an individual's ability to make sense of it. If we don't know how a checking account works, for example, we would need some background information before being able to proceed with this problem. Whenever we are unfamiliar with a context, it often helps to organize the data in a table. As we determine the data that belong in the table, we become more familiar with the specifics of the problem. Then we may represent the numerical information for each bank's plan as points on a graph.

The place at which the two points overlap (18 checks) is the break-even point—for this number of checks, each account costs the same amount. If a person writes fewer than 18 checks a month, Fast Bank's plan is more economical. If he or she writes more than 18 checks a month, Thrifty Bank's is.

NUMBER OF CHECKS	THRIFTY BANK COSTS	FAST BANK COSTS
0	4.50	0
2	5.00	1.00
4	5.50	2.00
6	6.00	3.00
8	6.50	4.00
10	7.00	5.00
12	7.50	6.00
14	8.00	7.00
16	8.50	8.00
18	9.00	9.00
20	9.50	10.00



We could also represent the problem with equations: Thrifty Bank, $y = .25x + 4.50$; Fast Bank, $y = .50x$, where x represents the number of checks and y represents the cost of the account in dollars. Setting the two equations equal to each other, we can then solve for x algebraically:

$$.50x = .25x + 4.50$$
$$.25x = 4.50$$
$$x = 18$$

Teaching Algebra

While algebra used to be a subject reserved only for students going to college or those interested in advanced study, today we realize that algebraic reasoning is central to many other subjects and vocations. School systems throughout the country are requiring that all students take algebra. The National Council of Teachers of Mathematics has recommended that algebra be a curricular strand in kindergarten through grade 12 mathematics (NCTM 2000). If students have had algebraic experiences before they encounter algebra as a course in middle or high school, they will more easily make the transition from reasoning about number to reasoning about symbols and relationships. In the early elementary grades, studying patterns is the most prominent algebraic experience; patterns are the basis for reasoning about regularity and consistency. As students move into the upper elementary and middle grades, they need to generalize these patterns and express the relationships in a variety of ways. Students learn to use language, tables, and graphs to represent relationships and to formalize them using function rules and equations. At all levels, it is important to ask students to justify their statements using pictures, examples, and reasoning.

Questions for Discussion

1. Examine a textbook and find examples of where students are asked to reason algebraically. Explain the mathematical purpose of the activities.
2. Compare and contrast how variables are used.
3. Some people think that algebra is too complex a topic to teach to young children. What do you think? Support your statements.
4. Patterns are a major topic in mathematics. Describe the different types of patterns and describe one instructional activity that you could use to have students explore each pattern type.