Activity 1.1: The World of Codes

The purpose of this activity is to help you decide what makes one coding process better than another.

In this activity, you play the role of code cracker and try to decode five messages. Each message is coded with a different process. You may not be able to crack all of them now. But you will become a better code cracker as you go through the chapter.

For now, try to find the meaning of as many of the messages as you can. Later you will have chances to revisit these messages and apply new techniques for cracking codes.

TAKE NOTE In this course, the problems you solve are as real as is possible. However, you would not be happy if the messages you had to crack took months of effort. So, the messages in this and other activities are short. They are not real historical messages, but they contain facts about the use of codes in the world.

Message 1 −8

BPWUIA RMNNMZAWV QVDMVBML I EPMMT TQSM LMDQKM BW KWLM UMAAIOMA

Message 2 −5

6 19 26 18 7 10 23 8 6 17 17 10 9 6 8 13 10 8 16 9 14 12 14 25 14 24 26 24 10 9 25 20 9 10 25 10 8 25 10 23 23 20 23 24 14 19 8 20 9 10 24 17 14 16 10 31 14 21 8 20 9 10 24

Message 3

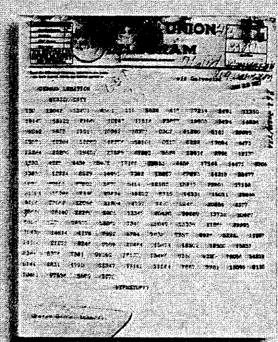
79 63 55 23 63 27 83 35 23 23 99 35 39 11 39 83 79 7 83 83 35 23 59 7 83 39 63 59 7 51 15 75 103 67 83 63 51 63 31 39 15 55 87 79 23 87 55 15 7 59 11 23 79 23 23 59 63 59 83 35 23 95 63 75 51 19 95 39 19 23 95 23 11

Message 4

HRKND JDNK GR UIBV EBHF HRLUBHG KZDFD BAN BHGJBIIV BEIN GR HRAANHG NAARAD

FYI

Secret codes have an important role in history. Military commands were coded during World Wars I and II. The countries with the best cryptographers held an advantage in both wars. The famous Zimmermann telegram was intercepted and decoded by the British. It revealed Germany's attempt to ally with Mexico against the United States in World War I. President Woodrow Wilson asked for a declaration of war shortly after this telegram was intercepted.



As you can tell by looking at the telegram, coded messages are often long. Cracking them can take months of effort by a team of cryptographers.

Message 5

37 2 29 29 27 23 24 14 14 38 14 21 24 15 28 39 21 23
24 12 18 34 12 27 37 21 17 25 14 36 38 24 23 32 6 32
30 22 10 27 10 34 41 16 29 25 8 38 28 5 27 21 9 27
38 4 24 33 18 39 37 10 12 21 25 27 38 15 28 29 19 41
38 19 21 24 28 19 41 10 18

1. Try to crack one or more of the messages. Prepare to present your results to the class.
2. Of the messages you cracked, which were the hardest? Give reasons for your answer.
3. What makes one coding model better than another?

Before you continue with your coding, decoding, and code cracking, you should know a few terms:

The original form of a message before it is coded is called the **plaintext**.

Code and **cipher** can have different meanings in some books, but they have the same meaning in this chapter. Both refer to the method used to encode a message.

The process of putting a message in coded form is called **encoding**. Taking it from coded form back to plaintext is called **decoding** or **deciphering**.

Finding the contents of a message without being told how it was coded is called **code breaking** or **code cracking**.

Cryptography refers to the study of coding, decoding, and code breaking.

Individual Work 1.1: Sending Information

In this Individual Work you look at different types of codes. You also explore three simple models for secret coding: the shift cipher, the substitution cipher, and the transposition cipher.

FYI

As you try to crack coded messages, you might keep in mind the words of Captain Parker Hitt, who wrote the first U.S. Army manual on code cracking: "Success in dealing with unknown ciphers is measured by these four things in the order named: perseverance, careful methods of analysis, intuition, and luck." This chapter provides chances to learn "careful methods of analysis." You will have to supply the perseverance.

1. Recall that not all codes are used for secrecy. Some allow information to be sent quickly and efficiently. For example, the U.S. Postal Service uses zip codes to speed the mail. ("Zip" stands for "zone improvement program.") The simplest zip code is a five-digit number. (Some have nine digits; others have eleven.)

   Zip codes are converted to a series of long and short bars that are read and sorted by a machine. The bars for several five-digit zip codes are shown in **Figure 1.1**.

   17011

   20078

   19006

   05445

   Figure 1.1.
   The bars for four zip codes.

   a) Find the code for each digit from 0 to 9.

   b) Why do you think the Postal Service uses five characters with three short bars and two long ones? Could they have used two short bars and three long bars instead?

   c) If the Postal Service used six bars with one long and five shorts in each group, how many numerals could be coded?

   d) With six bars, how many longs and shorts would have to be used to cover ten numerals?

   e) A good secret code is easy to encode, easy to decode, and hard to crack. Do you think that zip-code bars would make a good secret code?

2. Zip codes and universal product codes (UPCs) are common types of codes. Name other situations in which codes are used.

3. a) Write a short message. Design a secret code and use it to encode your message. Explain or diagram the coding process you used.

   b) Challenge a member of your family or a friend to decipher your coded message. Describe the success or failure of the person who tried to crack your code. Note how long it took for the person to crack the code or to give up.

4. Julius Caesar was the emperor of the Roman Empire during the first century B.C. Caesar used a simple coding process called a shift cipher that replaces each letter of the alphabet with a letter three places removed.

A model for coding that uses addition or subtraction to move or shift every letter of the alphabet to another letter in the alphabet is called a **shift cipher**. For example, a shift of +5 codes the letter *C* as the letter *H* because *H* is five letters beyond *C*.

a) How does the shift (+5) cipher code the letters *M* and *Y*?

b) Use a shift (+8) cipher to code the first five words of this message: "Benedict Arnold used a substitution cipher to send messages to the British in the Revolutionary War."

5. For parts (a)–(d), use the given shift cipher to code the given letter.

a) shift (+5); Q b) shift (+11); W

c) shift (–8); N d) shift (+26); R

6. Another coding model is a **substitution cipher**. It replaces words or letters with other words or letters. The telegram in **Figure 1.2** is an example. It was sent to a Colonel Pelton about the 1876 presidential race. This message was coded by using replacements for certain sensitive words.

The coded message reads: Certificate required to Moses decision have London hour for Bolivia of just and Edinburgh at Moselle hand a any over Glasgow France received Russia of.

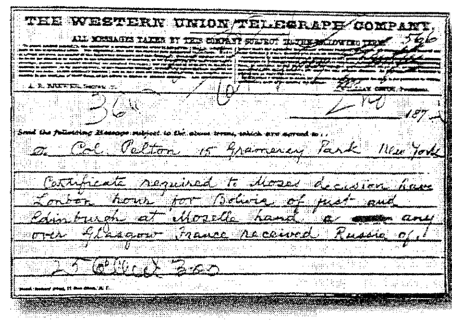


Figure 1.2. The Pelton telegram.

a) Make the following substitutions and write the message that results: replace *Bolivia* with *proposition*, *Russia* with *Tilden*, *London* with *canvassing board*, France with *Governor Stearns*, *Moselle* with *two*, *Glasgow* with *hundred*, *Edinburgh* with *thousand*, and *Moses* with *Manton Marble*.

b) The message in part (a) still doesn't make sense because it was also coded with a transposition cipher. In a transposition cipher, the order of the words or letters is changed.

The list below can help you unscramble the message. For example, the first word of the original message is the eighteenth word of the coded message, the second word of the original message is the twelfth word of the coded message, etc. (Hint: Governor Stearns, Manton Marble, and canvassing board each count as single words.)

18, 12, 6, 25, 14, 1, 16, 11, 21, 5, 19, 2, 17, 24, 9, 22, 7, 4, 10, 8, 23, 20, 3, 13, 15

Write the unscrambled message.

FYI

The 1876 election between Rutherford Hayes and Samuel Tilden was one of only three times in U.S. history that a candidate won the popular vote but lost in the Electoral College. The telegram had an offer to sell Florida's electoral votes and thereby swing the 1876 election from Hayes to Tilden.

Activity 1.2: Move Over and Stretch

In this activity, you encode messages with shift ciphers and stretch ciphers. You also use tables and arrow diagrams to describe the coding process.

In a shift cipher, every letter in the message is replaced by a letter that is a fixed number of places from it in the alphabet. Another way to do the coding begins by giving a position number to every letter.

Each letter has a **position number** from 1–26 based on its place in the alphabet, as shown in **Figure 1.3.** The **coded value** of the letter is a new number that results from applying a coding process.

Plaintext letter	Position number	Plaintext letter	Position number
A	1	N	14
B	2	O	15
C	3	P	16
D	4	Q	17
E	5	R	18
F	6	S	19
G	7	T	20
H	8	U	21
I	9	V	22
J	10	W	23
K	11	X	24
L	12	Y	25
M	13	Z	26

Figure 1.3.
The position numbers of the letters of the alphabet.

The position number of K is 11. A shift of +6 results in a coded value of 11 + 6 = 17. If the coded value is changed to a letter, the original K becomes a Q.

1. a) Code the sentence "I know a secret" into a text message by using a shift of –3.

   b) Code the same sentence into a numerical code. Replace each plaintext letter with its position number. Then use a shift of –3 to change the position numbers to coded values.

2. Suppose you shift the alphabet more than 26 spaces. For example, use a coding process that shifts the alphabet +30. How do you code a letter as another letter?

3. Coding models can be represented in a variety of ways. **Figure 1.4** shows an **arrow diagram** that represents a coding process.

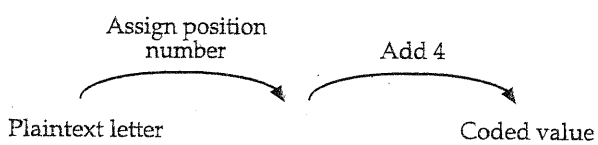


Figure 1.4.
Shift +4 arrow diagram.

   a) Describe the coding process.

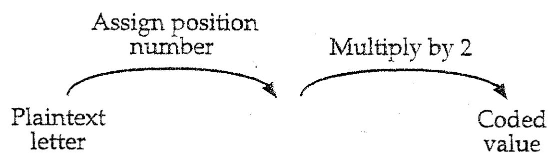
   b) Draw a new arrow diagram representing the same process, but with letters coded as letters.

c) Use an arrow diagram to show a shift of +3 followed by a shift of +7. Copy **Figure 1.5** onto your paper and write the description of each step above its arrow.

Figure 1.5. A three-step arrow diagram.

d) What are some advantages of this representation? That is, would the coder, decoder, or code breaker find it useful?

4. The arrow diagram in **Figure 1.6** shows a new coding process. The table is a partial representation of the same process.



Plaintext letter	A	B	C	D	E	F	G
Position number	1	2	3	4	5	6	7
Coded value	2	4	6	8	10	12	14
Plaintext letter	H	I	J	K	L	M	
Position number	8	9	10	11	12	13	
Coded value	16	18	20	22	24	26	

Figure 1.6. Arrow diagram and table for a new coding process.

a) Use this process to encode the word *elastic*.

b) Why is this cipher sometimes called a **stretch cipher**?

c) Draw an arrow diagram to represent a coding process that stretches the distance between coded values by more than 2. Use this process to code the word *elastic*.

d) A stretch cipher was used to encode the word *stretch* as 76 80 72 20 80 12 32. Describe the coding process.

When a message is coded, a plaintext letter or position number is matched with a unique coded value or letter. Such a matching process is used in mathematics for tasks other than coding.

> A **function** is a process that transforms items such as letters or numbers into other letters or numbers uniquely.

A function produces a single output from an input. In the context of codes, the coded value is the output. It results from applying the coding process to the original position number. The original position number is the input. In the context of a business, the input may be the number of items sold and the output may be the total profit.

TAKE NOTE An arrow diagram is a good way to show the steps of a process like a function. A table is another way to represent a function. A table lists some or all of the matched pairs of inputs and outputs.

5. The table in **Figure 1.7** represents part of a coding process. Does the table represent a function? Explain.

Position number	1	2	3	4	5
Coded value	6	12	18	24	30

Figure 1.7.

6. The table in **Figure 1.8** shows the ages and heights of the students in a school's math club. Does the table represent a function? Explain.

Age (years)	16	15	18	16	17	14	17	18
Height (inches)	63	56	68	66	61	60	66	73

Figure 1.8.

7. The arrow diagram in **Figure 1.9** represents a shift coding process.

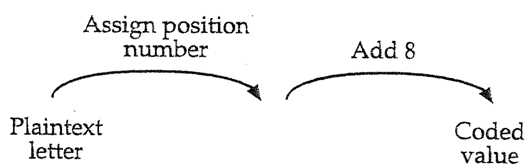


Figure 1.9.
An arrow diagram for a coding process.

a) Use the arrow diagram in Figure 1.9 to complete the table in **Figure 1.10.**

Plaintext letter	A	B	C	D	E	F	G	H	I	J	K	L	M
Original position													
Coded value													
Plaintext letter	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Original position													
Coded value													

Figure 1.10.

b) What are some advantages of the table? That is, would the coder, decoder, or code breaker find it useful?

8. You have used addition, subtraction, and multiplication to encode messages. Are there any numbers that can't be used in these coding processes? Explain.

9. Is it possible to design a coding process that matches a letter to the number 4.5?

People who exchange coded messages must tell each other how they are coding so that the message can be decoded. Sharing the method is a classic problem of secret codes. Telling someone you are coding by "adding 3" requires a brief communication; telling someone you are coding by an entire table requires a larger one.

People who code and decode often use computers. Computers and calculators work better with numbers than with letters.

Symbolic forms, which are the topic of Activity 3, are used by computers and graphing calculators. They allow the calculator to do much of the work. Symbolic forms are easy to change. Coders often change their process in order to foil code crackers.

Activity Summary

In this activity, you:

- used shift ciphers and stretch ciphers to encode messages.
- represented the coding process with arrow diagrams and tables.
- interpreted the coding process as a function.

DISCUSSION/REFLECTION

1. Recall from the Preparation Reading that a good coding model uses a coding process that is simple and is easy to share. Do shift ciphers and stretch ciphers provide good coding models?
2. In your coding, you left spaces between words alone. What are some ways to include spaces in the coding?
3. Why is it better to code a letter as a number rather than as a letter?

Individual Work 1.2: Arrows and Graphs

You have used shift ciphers and stretch ciphers to encode messages. In this Individual Work you learn to use graphs to represent coding processes. You also practice coding and using arrow diagrams to represent the coding process.

1 a) Write a message of no more than ten words.

b) Select a number of letters to shift the alphabet. Make a table like the one in **Figure 1.11** for your shift.

Plaintext letter	A	B	C	D	E	F	G	H	I	J	K	L	M
Code letter													
Plaintext letter	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Code letter													

Figure 1.11.

c) Use the table to code your message in letters.

d) Code your message in numbers.

2. Code your message from Question 1 using a stretch cipher.

3. Suppose you code by shifting the alphabet +4 followed by a shift of –7.

   a) This is a two-step process. Can you shorten it to one step? If so, how? If not, why not?

   b) If you shift by –7 before you shift by +4, are the coded values different? Explain.

4. Coders often change their method in case a code breaker has cracked the code. Would changing from a shift (+20) to a shift (–6) foil a code breaker who has cracked the shift (+20) cipher? Explain.

5. Scientists use two scales to record temperature: Celsius and Kelvin. You can convert Celsius to Kelvin by adding 273. For example, 30°C is 303 K. (The degree symbol is not used with Kelvin.)

   a) Convert 83°C to Kelvin.

   b) Convert –20°C to Kelvin.

   c) Explain how to change Kelvin to Celsius.

   d) How is changing from Celsius to Kelvin like using a shift cipher?

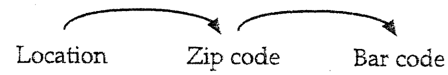
FYI

The Celsius scale is sometimes called the *centigrade* scale. This is because it was made by dividing the temperature range between the freezing point and boiling point of water into 100 parts. The Kelvin scale is sometimes called the *absolute temperature* scale. A temperature of 0 K is known as absolute zero. There is no temperature lower than 0 K.

6. An arrow diagram is a visual representation of a process.

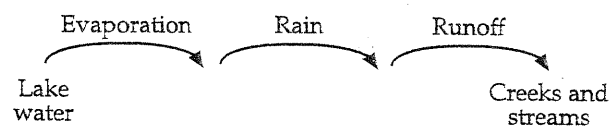
   a) Explain the process in **Figure 1.12** in words.

Figure 1.12.



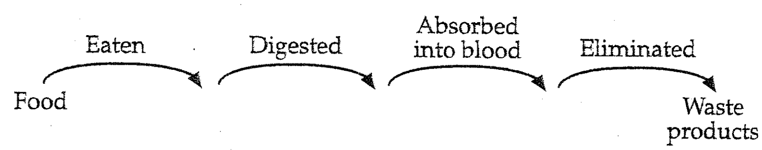
   b) Explain the process in Figure 1.13 in words.

Figure 1.13.



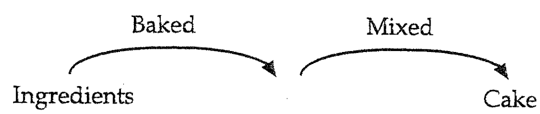
   c) Explain the process in **Figure 1.14** in words.

Figure 1.14.



7. Describe what is wrong with the arrow diagram in **Figure 1.15.**

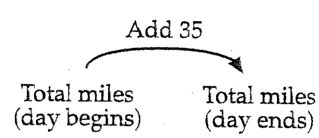
Figure 1.15.



8. An arrow diagram can be used to represent a mathematical process. Suppose that each week you get an allowance of \$15.00. From that \$15.00 you must pay \$5.00 for your gym class. Draw an arrow diagram.

9. Arrow diagrams describe processes. A table shows how symbols and values are paired. Use the arrow diagram to complete the table in **Figure 1.16.**

Figure 1.16.
An arrow diagram and the related table.



Day	0	1	2	3	4	5	6	7
Total miles	0	35						

The following information is for Questions 10–18.

A coordinate graph is another way to represent a coding process.

A **coordinate graph** has two lines that intersect at right angles. Each line is called an **axis**. The **horizontal axis** is related to a function's input. The **vertical axis** is related to a function's output. The place where the two axes cross is called the **origin** (see **Figure 1.17**).

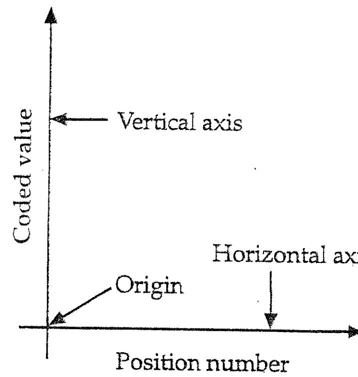


Figure 1.17. A coordinate graph for coding.

In this chapter, the horizontal axis represents the plaintext letter or its position number. The vertical axis represents the coded value or the coded letter.

Every point on a graph is identified with an **ordered pair** of numbers called **coordinates**. The first number represents horizontal position and the second number represents vertical position. For example, the coordinates (5, 8) identify the point on the graph with a horizontal position of 5 and a vertical position of 8 (see **Figure 1.18**).

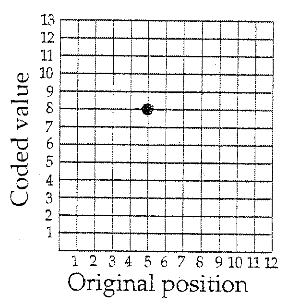


Figure 1.18. The graph of (5, 8).

In the context of codes, a point's coordinates connect a position number with a coded value. The point with coordinates (5, 8) means the letter with original position number of 5 (the input) is matched with the coded value of 8 (the output).

Note that in Figure 1.18 the description is "The graph of (5, 8)." Both axes are numbered with scales of one unit for each square. The vertical axis is labeled Coded value, and the horizontal axis is labeled Position number.

When no context is stated, the variables x and y are used to denote the horizontal and vertical coordinates.

10. For parts (a) and (b), plot the ordered pairs on a coordinate graph.

    a) (3, 9), (1, 6), (4, 2), (7, 6), (5, 0)

    b) (–2, 5), (3, 2), (–3, –1), (4, –2)

TAKE NOTE

A good graph has the following traits:

- The graph's contents are explained with a title or brief description.
- Each axis is numbered with a consistent scale.
- Each axis has a label.

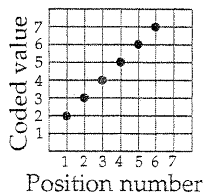


Figure 1.19.
Part of the graph of a shift cipher.

11. The graph in **Figure 1.19** shows the way a shift cipher codes the first six letters of the alphabet.

    a) Describe the shift cipher in words.

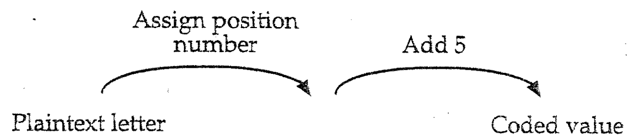
    b) Does this shift cipher code the letter H as a letter or as a number?

    c) Use this cipher to code the word *codes*.

    d) The word 17 19 16 4 6 20 20 was coded with this shift cipher. Decode it.

12. The arrow diagram in **Figure 1.20** represents a coding process.

Figure 1.20.



   a) Use the arrow diagram to complete the table in **Figure 1.21**.

Plaintext letter	A	B	C	D	E	F
Position number	1	2	3	4	5	6
Coded value						

Figure 1.21.

   b) Use a sheet of graph or dot paper (as in **Figure 1.22**) to make a graph that represents the coding process for all 26 letters of the alphabet.

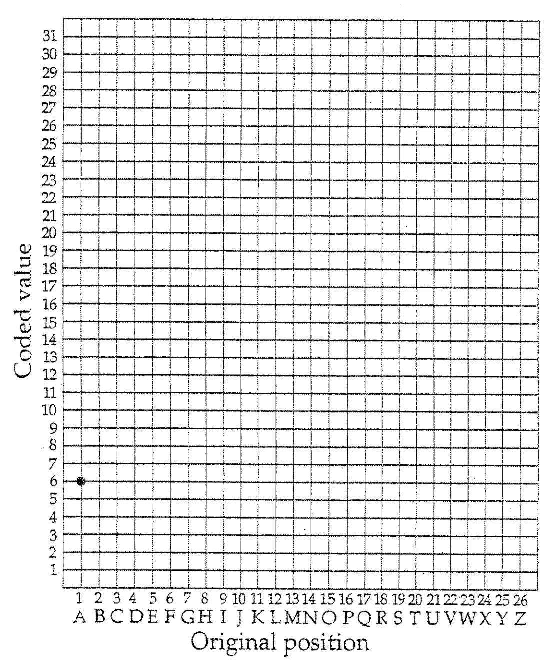


Figure 1.22.
Graph of coded value versus original position.

c) Describe your graph.

d) How is the graph different if you shift eight instead of five?

e) What are some advantages of this representation? That is, would the coder, decoder, or code breaker find it useful?

13. The graph in **Figure 1.23** represents part of a coding process.

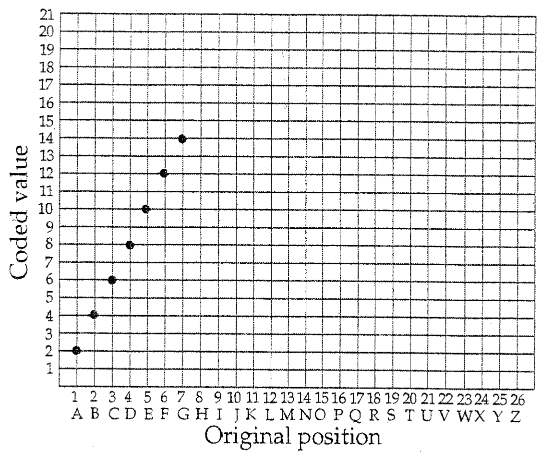


Figure 1.23.

a) Describe how to decide where to draw the next two dots on the graph.

b) What is the meaning of the point (8, 16)?

c) Use the graph to encode the word *king*.

d) The letter C is coded as the number 6. The letter E is coded as the number 10. Which letter is coded as the number 26?

14. a) How would you share the coding process of Question 13 with another person without using the graph?

b) Explain why the number 15 cannot represent a letter.

c) Explain how the graph of a stretch cipher differs from the graph of a shift cipher.

15. If you want a friend to decode your messages, you must share the coding process. You can use an arrow diagram, a table, or a graph to do so. Which would you choose, and why?

16. The table in **Figure 1.24** shows the cost of sending a package based on its weight.

Weight (in pounds)	Total shipping cost ($)
1	3.45
5	7.15
10	11.00
15	14.44
20	16.35

Figure 1.24. Shipping cost.

a) What does it cost to send a 14-pound package?

b) What does a package weigh if it costs $8.00 to ship?

17. The graph in **Figure 1.25** represents the cost of renting a power tool. The cost depends on the number of hours the tool is rented.

a) Use the graph to estimate the cost of renting the tool for 9 hours.

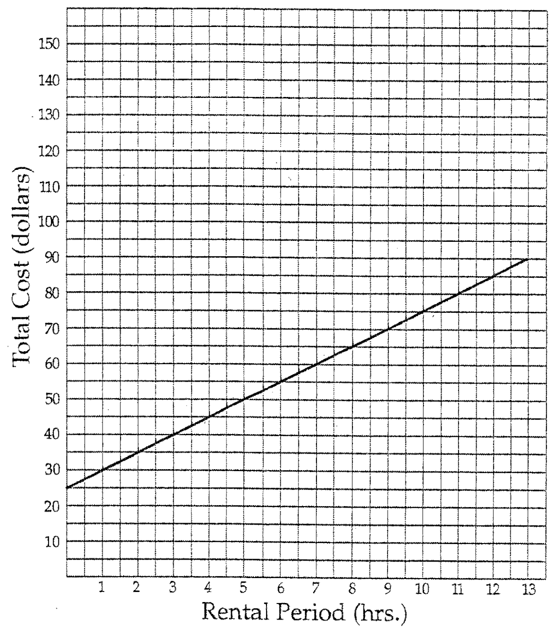


Figure 1.25.
Equipment-rental cost.

b) About how many hours did you rent the tool if the cost is $55.00?

18. Find a table or graph in a book, magazine, or newspaper. Draw, photocopy, or cut out the table or graph. Bring it to class to share. Write a sentence or two explaining the information that the table or graph conveys to the reader.

Activity 1.3: Calculator Coding

Graphs can be drawn on paper. They can also be made with calculators or computers. In this activity, you learn to use a graphing calculator to encode secret messages.

To use a graphing calculator to encode a message, you must first translate a coding process into a symbolic equation.

An **equation** is a mathematical relationship that states that two quantities are equal. A **symbolic equation** is an equation that uses only symbols.

For example, the arrow diagram in **Figure 1.26** represents a positive shift of three units. You add 3 to the original position number to find the coded value. This statement can be written in the form of a word equation:

coded value = position number + 3.

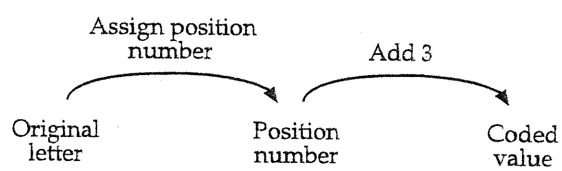


Figure 1.26. A coding process.

To keep equations brief, a single letter is often used to represent a word or phrase. Make the following replacements:

- Use the letter c as a coded value
- Use the letter p as a position number

Now the equation can be written in symbolic form:

$$c = p + 3.$$

Names are commonly given to the parts of symbolic equations.

A **variable** is a quantity that changes.

The letters c and p are both variables in the equation $c = p + 3$ because they represent numbers that change depending on the letter being coded.

A **constant** remains unchanged whenever a given equation is used. A constant is sometimes called a **control number**. In a coding process, a constant "controls" the amount of a shift or stretch.

In the equation $c = p + 3$, the number 3 is a constant. In the equation $c = 2p$, the number 2 is a constant.

1. a) Write a word equation and a symbolic equation to represent a shift of +12.

   b) Write a word equation and a symbolic equation for the arrow diagram in Figure 1.27.

FYI

A constant that multiplies a variable is called a ***coefficient***. In the equation $c = 2p$, 2 is the coefficient of p. In the equation $c = p + 3$, the coefficient of p is 1.

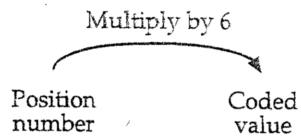


Figure 1.27.

2. For (a)–(d), describe the coding process that each equation represents.

   a) $c = p - 7$

   b) $c = 4p$

   c) $c = p + 20$

   d) $c = -3p$

TAKE NOTE

The equations $c = p + 3$ and $Y = X + 3$ describe the same function. Recall that a function transforms a number into another number uniquely. In these equations, the symbols c, p, Y, and X represent variables. The corresponding tables, graphs, and arrow diagrams for the two equations are identical.

Equations can be built using any letters you choose for the variables. But most graphing calculators use just X and Y when graphing.

- The variable X represents the input. For a coding process, this is the position number of the plaintext letter.
- The variable Y represents the output, which is the coded value.

The symbolic equation $c = p + 3$ becomes $Y = X + 3$ when a graphing calculator is used.

Many calculators can draw graphs of several equations at once. These calculators attach a number to Y so that the equations can be given distinct labels. The first equation would be $Y_1 = X + 3$. A second equation, might be $Y_2 = X + 10$.

FYI

The numbers 1 and 2 as used in the symbols Y_1 and Y_2 are called ***subscripts***. Subscripts are often used to distinguish two quantities that are represented with the same letter symbol.

To make a graph on a calculator, you need to do two things:

- Enter the equation. On most calculators, only the side of the equation containing the input (X) is typed. The Y_1 or Y_2 is already displayed on the screen.
- Set the window. Enter the minimum and maximum values for X and Y. Choose these values the same way you would choose the scales for a graph you draw on paper. It is helpful to choose a window slightly larger than the values you want to graph. And it is a good idea to include the coordinate axes. This means including the value 0 for both variables in the window.

3. The equation $c = p + 3$ can be entered into a graphing calculator as $Y_1 = X + 3$. Since X represents the position number, the smallest value for X is 1 (for A) and the largest is 26 (for Z). When you code using a shift (+3) cipher, the smallest coded value is 4 (for A) and the largest value is 29 (for Z).

   a) Enter the equation $c = p + 3$ in the graphing calculator as $Y_1 = X + 3$. Set your window to Xmin = 0, Xmax = 30, Ymin = 0, and Ymax = 30.

   b) Graph the equation. Sketch the screen on your paper.

TAKE NOTE

When you want to sketch a graph on paper from a calculator screen, you should record the minimum and maximum values for the window. They should be enclosed in brackets. For example, [0, 30] x [5, 40] means that Xmin is 0, Xmax is 30, Ymin is 5, and Ymax is 40. **Figure 1.28** is a sample sketch of the equation $Y_1 = 2X$.

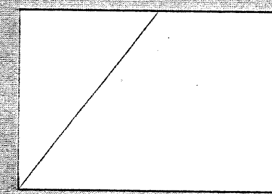


Figure 1.28.
Sketch of a calculator graph of $Y_1 = 2X$.

$Y_1 = X + 3$ and $Y_1 = 2X$ describe **linear functions**. They are linear because their graphs are straight lines. They are functions because for each X or original position number (input), there is exactly one Y or coded value (output).

Special terms are used to describe all of the possible input and output values for a function.

- The **domain** is the collection of all the x-values (inputs).
- The **range** is the collection of all the y-values (outputs).

The function pairs each number from the domain with exactly one number from the range.

FYI

The graph of a coding process looks different when made with technology than when drawn on paper. On paper, you can plot the separate points that represent (position number, coded value) pairs. Such a graph has a series of discrete points. Calculator graphs look continuous; that is, points seem to be connected.

A sidewalk can be thought of as a continuous path. A path made of stepping stones is discrete.

A steady stream of water is a continuous flow. A dripping faucet is a discrete flow.

The domain of a linear function like $y = x + 3$ is all real numbers. This is because 3 can be added to any real number. The range of the function is also all real numbers because any real number can be the result of adding 3 to some other number.

But for most of the coding processes in this chapter, only input values from 1 through 26 make sense. Thus, the domain for this situation is different from the domain of the mathematical function:

Domain of the function $y = x + 3$: all real numbers;

Domain of the coding process $c = p + 3$: integers from 1 through 26.

When a mathematical function is used to model a real situation, the context may require a domain that is different from the domain of the function. In the context of a given model, the domain of the situation is often a smaller set of numbers than the domain of the function.

FYI

Some graphing calculators use range to describe the largest and smallest values for both X and Y that appear on the graphing screen. Other calculators use the term window for the same purpose. The term window is used in Mathematics: Modeling Our World.

For a coding process, the range is the collection of coded values and varies with the coding process used.

4. Now that you have graphed the equation in Question 3, you will use it to encode letters. For example, to encode P, start with its position number, 16. Use the trace feature of the calculator to trace to the point on the graph where X is as close as possible to 16. The Y-value should be very close to the coded value. Use the graph from Question 3 to find the coded value for P.

5. Use the calculator and your graph from Question 3 to encode the word *window*.

6. Some graphing calculators have a table feature. The table must be set up before it can be displayed. For this table, input values should start at 1 and increase by units of 1.

   a) Make a table for the coding process $c = p + 3$. When you have the table you want, scroll up and down to see more of it.

   b) Use the table to encode the word *range*.

7. The graphing calculator allows you to graph more than one equation at a time. Keep the first equation you entered in Question 3 as $Y_1 = X + 3$.

   a) Write a function to represent a coding process with a shift of 12. Use Y_2 for the output variable.

   b) Adjust Ymax in the window to include all the values related to a shift (+12) cipher.

   c) Enter the equation for Y_2 in the calculator, and graph it. You should see two lines on the screen, one for Y_1 and one for Y_2. Draw a sketch of the screen on your paper. Label each line with its equation.

   d) Use the calculator and a shift (+12) cipher to encode the word *crypto*.

TAKE NOTE

It can be hard to get a graphing calculator to trace to the exact point you want. For example, **Figure 1.29** is a calculator graph of $Y_1 = X + 3$ traced as close as possible to the point where X is 10. The readouts of 9.893617 for X and 12.893617 for Y are the coordinates of the pixel where the trace is stopped.

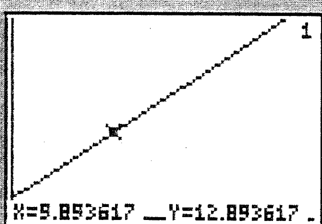


Figure 1.29. Calculator graph of $c = p + 3$ or $Y_1 = X + 3$.

If this graph represents a coding process, then the best you can do is to code J, the tenth letter of the alphabet, as 13. You know that J codes as a whole number, so it is reasonable to give the coded value as 13.

FYI

To set up a calculator table, the starting input value and the difference between successive input values must be entered. Some calculators use ΔX or ΔTable to represent the difference in input values. Δ is the Greek letter delta.

MODELING NOTE

One way modelers use technology is to make tables. Tables can be created with calculators, computer spreadsheets, or other software. Factors that influence a modeler's choice of technology include:

It must be readily available.

The modeler must be familiar with it.

It must have the features needed for the task.

8. a) Describe how the graph for $Y_1 = X + 3$ differs from the graph for $Y_2 = X + 12$.

   b) Describe how the graph for $Y_1 = X + 3$ is similar to the graph for $Y_2 = X + 12$.

   c) Examine some graphs for other shift ciphers. In general, what similarities and differences do you see?

   d) What type of coding process might result in a line that is not parallel to $Y_1 = X + 3$?

9. a) Describe how the graph for $Y_1 = 2X$ differs from the graph for $Y_2 = 5X$.

   b) Describe how the graph for $Y_1 = 2X$ is similar to the graph for $Y_2 = 5X$.

   c) Look at some graphs for other stretch ciphers. In general, what similarities and differences do you see?

Activity Summary

In this activity, you:

- used symbolic equations to represent coding processes.
- used a graphing calculator to make tables and graphs.

DISCUSSION/REFLECTION

1. How can you recognize a shift cipher from its graph?
2. How can you recognize a stretch cipher from its graph?
3. If a point is on the calculator graph of a coding process, do the point's coordinates always represent a (position number, coded value) pair? Explain.
4. Tables, graphs, equations, arrow diagrams, and graphs or tables on a calculator are methods that you can use to code messages. List advantages of each representation.

Individual Work 1.3: Coding with Graphs

In this Individual Work, you practice using equations and graphs to represent coding processes. You also look at another way to use a calculator for coding.

To use a graphing calculator for coding, you must write the process in a symbolic form.

For Questions 1–4,

a) Write a word equation for the given arrow diagram.

b) Convert your word equation to a symbolic equation.

c) Write your equation in a form that can be used on a calculator.

1. See **Figure 1.30.**

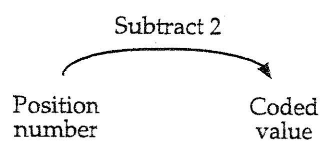


Figure 1.30.

2. See **Figure 1.31.**

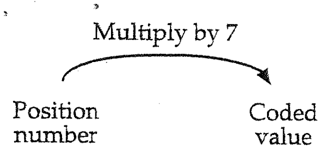


Figure 1.31.

3. See **Figure 1.32.**

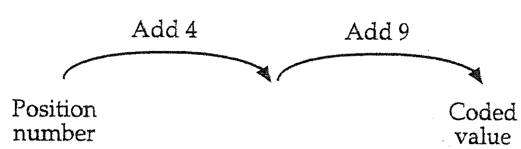


Figure 1.32.

4. See **Figure 1.33.**

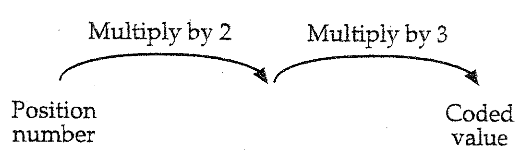


Figure 1.33.

For Questions 5–6,

a) Change the symbolic equation to a word equation.

b) Draw an arrow diagram for the equation.

c) Describe the coding process represented by the equation.

12. Compare a coding process that adds 3 with one that multiplies by 3.

    a) What are the smallest and largest coded values for the addition process?

    b) State appropriate values for *Y*min and *Y*max for the addition process.

    c) What are the smallest and largest coded values for the multiplication process?

    d) State appropriate values for *Y*min and *Y*max for the multiplication process.

13. a) Use a graphing calculator to graph a shift (+3) cipher and a stretch (+3) cipher on the same graph.

    b) What window did you use for this graph?

    c) Describe how the graphs are alike.

    d) Describe how the graphs differ.

    e) How can you use the graphs to tell if a shift (+3) cipher and a stretch (+3) cipher code any letters the same?

14. **Figure 1.35** shows four ways to represent the process of computing a 6% sales tax.

GRAPH

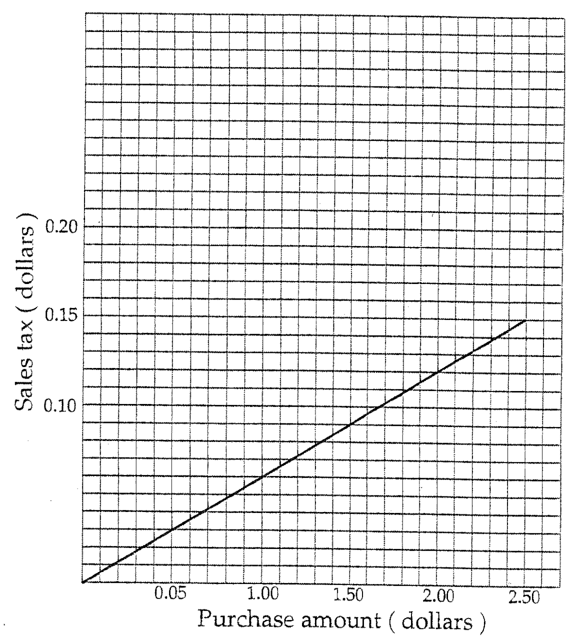
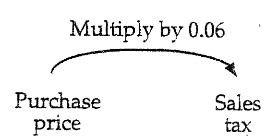


Figure 1.35.
Four ways to represent a 6% sales tax.

TABLE

Price	Tax
0.50	0.03
1.00	0.06
1.50	0.09
2.00	0.12
2.50	0.15

ARROW DIAGRAM



EQUATION

$$\text{Tax} = 0.06 \times \text{Price}$$

5. $c = -10p$

6. $c = p + 5$

7. The graph for a shift (+5) cipher is not the same as the graph for a shift (+3) cipher.

   a) Explain how the graphs are alike.

   b) Explain how the graphs differ.

8. Suppose you code by shifting the alphabet –8.

   a) Draw an arrow diagram for this coding process.

   b) How do you code the letter B as a number and as a letter?

   c) Write the equation you would enter into a graphing calculator.

   d) The position number for the letter A is 1 and for Z is 26. The calculator window must have Xmin = 1 or less and Xmax = 26 or greater. For the shift (–8) cipher, state values for Ymin and Ymax.

9. Suppose you code by stretching the alphabet +4.

   a) Draw an arrow diagram to represent this coding process.

   b) Code the letter K as a number.

   c) Code the letter K as a letter.

   d) Write the equation you would enter into a graphing calculator.

   e) For the stretch (+4) cipher, state values for Ymin and Ymax.

10. a) Write a symbolic equation for a stretch (+2) cipher.

   b) Does this process represent a linear function? Explain.

   c) How does the graph of the stretch cipher differ from the graph of a shift cipher like $c = p + 2$?

11. *Distance = (rate)(time)* is an equation used to find distance, given the rate at which an object travels and the time of travel. If a car travels at a constant rate of 60 mph, the equation is $d = 60t$, where d is miles and t is hours. The arrow diagram in **Figure 1.34** shows this relationship.

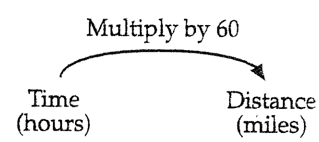


Figure 1.34.
Time and distance arrow diagram.

   a) For a trip that lasts 4 hours, what is the range of the distance function?

   b) Graph this equation for all trips that last up to 4 hours.

   c) Find d when $t = 3$ hours.

   d) Find t when $d = 130$ miles.

   e) How does the graph differ if the speed is 50 mph?

State which one of these representations you would use to do each of these. Explain your choice.

a) Find the tax on $2.00.

b) Find the tax on $587.16.

c) Show that tax increases as the price increases.

d) Show that tax is found by multiplying and not by adding.

e) Compare a 6% sales tax with an 8% sales tax.

15. Compare the five representations: word description, arrow diagram, symbolic equation, table, and graph. Which do you prefer for each of these?

a) To code a word or message

b) To transfer the coding process to a graphing calculator

c) To share your coding process with someone else

d) To compare two different coding processes to see if they code one or more letters the same

16. a) What coding process reverses the alphabet? That is, what is the equation that codes A as Z, B as Y, C as X, etc.?

b) Is this a shift cipher?

c) Is any letter left unchanged by this coding process?

d) How does the graph of this process differ from the graph of a shift cipher like $c = p + 2$?

e) How is the graph of this coding process similar to the graph of a shift cipher like $c = p + 2$?

17. You have used tables to transform position numbers into coded values. A table can be written in a form called a **matrix**. For example, the numbers in the sales tax table in Figure 1.35 can be placed in a 5×2 matrix:

$$\begin{bmatrix} 0.50 & 0.03 \\ 1.00 & 0.06 \\ 1.50 & 0.09 \\ 2.00 & 0.12 \\ 2.50 & 0.15 \end{bmatrix}$$

The **dimensions** (or size) of a matrix are stated by writing the number of rows followed by the number of columns. Each of these is an example of a matrix:

$$\begin{bmatrix} 4 & 2 & 17 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -2 \\ 5 & 7 & 3 \\ 12 & -4 & 0 \end{bmatrix} \qquad \begin{bmatrix} 12 & 8 & 15 & 10 \\ 5 & 16 & 27 & 21 \end{bmatrix}$$

1 × 5 matrix   3 × 3 matrix   2 × 4 matrix

Two matrices can be added if they have the same dimensions. To add two matrices, add the corresponding members. For example,

$$\begin{bmatrix} 2 & -1 & 5 \\ 3 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 1 & -3 \\ 7 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 2+4 & (-1)+1 & 5+(-3) \\ 3+7 & 2+(-5) & 0+1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 2 \\ 10 & -3 & 1 \end{bmatrix}$$

You can use matrix addition to speed the process of coding a message with a shift cipher. You can do matrix addition on a calculator. These directions apply to most graphing calculators.

a) Use the calculator to code the word *matrix* with a shift (+5) cipher:
   - Convert the word matrix to position numbers: 13 1 20 18 9 24.
   - Enter the numbers into the calculator as a 1 × 6 matrix.
   - Name the matrix [A].
   - Build another 1 × 6 matrix consisting of all 5s. Name it [B].
   - Perform the matrix addition [A] + [B] and write the coded values

b) Use the calculator's matrix features to code *calculator* with a shift (–8) cipher.

Activity 1.4: Codes in Reverse

In this activity, you use arrow diagrams, tables, graphs, and equations to decode messages. You also compare the usefulness of each of these representations in the decoding process.

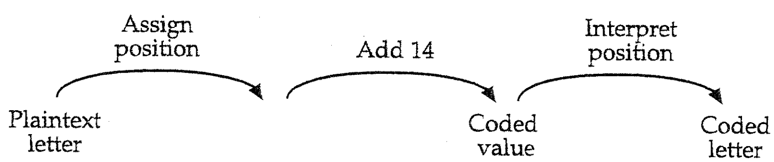
Recall that a coding model consists of a coding process and a decoding process. In Lessons 1 and 2, you found that a coding model must be easy to code and decode. Also it must be hard to crack.

1. In order to decode each of these phrases, you must reverse the process used to code them. Decode each of these messages:

   a) This phrase was coded with the process in **Figure 1.36.**

   S B W U A O Q C R S

Figure 1.36.



   b) This phrase was coded with the process in **Figure 1.37.**

   15 24 30 27 17 30 23 13 27 14 13 11 12

Figure 1.37.

Plaintext letter	A	B	C	D	E	F	G	H	I	J	K	L	M
Position number	1	2	3	4	5	6	7	8	9	10	11	12	13
Coded value							16						
Plaintext letter	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Position number	14	15	16	17	18	19	20	21	22	23	24	25	26
Coded value		24											

   c) This phrase was coded with the process in **Figure 1.38.**

   8 20 18 21 26 25 10 23 8 20 9 10 24

   d) This phrase was coded with the process $c = 3p$.

   15 36 15 66 15 42 12 27 21 27 60 78 27 48 9 45 12 15 57

2. Explain how you used the arrow diagram, the table, the graph, and the equation from Question 1 to decode the phrases.

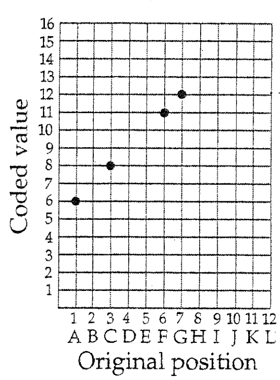


Figure 1.38.

The term **inverse** can be used to describe a process that converts a coded value back to its position number. For each message you decoded in Question 1, you used the inverse of the process that was used to code it.

3. Which representation worked best for decoding in Question 1: the arrow diagram, the table, the graph, or the equation? Explain.

4. Suppose a coding process replaces a letter with its position number and adds 3. The decoding process must first subtract 3 and then find the plaintext letter. The steps are reversed, and the inverse operations are used (see **Figure 1.39**).

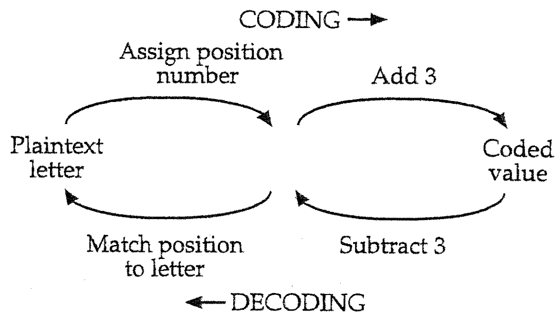


Figure 1.39.
An arrow diagram for both coding and decoding.

   a) Use the arrow diagram to encode the letter K.

   b) Use the arrow diagram to decode the number 15.

   c) What equation are you solving when you decode 15?

5. a) **Figure 1.40** is an arrow diagram of an encoding process. Draw an arrow diagram for the decoding process.

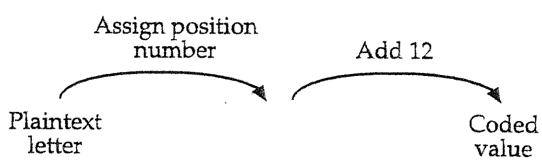


Figure 1.40.
Coding by adding 12.

   b) Suppose you encode by subtracting 8. Describe or diagram the decoding process.

   c) Suppose you encode by multiplying by 3. Write a decoding equation.

Activity Summary

In this activity, you:

- used arrow diagrams, tables, graphs, and equations to decode messages.
- compared the usefulness of representations of the decoding process.

Individual Work 1.4: Reverse that Process

In this Individual Work, you use inverse operations to solve equations and to find decoding equations. You also practice using arrow diagrams, tables, and graphs to decode.

1. Consider the coding process $c = p + 20$.
   a) Draw an arrow diagram of the coding and decoding processes.
   b) Explain how to use your arrow diagram to decode the coded value 30.

Another way to decode 30 is to **solve** the equation $30 = p + 20$. Your arrow diagram from (a) gives you a clue as to how to solve this equation.

A **solution** to an equation with one variable is any value for the variable that makes the equation true.

TAKE NOTE

The equations $30 = p + 20$, $30 - 20 = p - 20$, and $10 = p$ are **equivalent**. This means they have the same solution.

Performing the same operation on both sides of an equation results in another equation that is equivalent to the original one. Any of the four basic operations can be used to produce equivalent equations. But to solve a particular equation, you must choose the right operation. Understanding arrow diagrams and inverse operations can help you make the right choice.

When solving the equation $30 = p + 20$, your goal is to isolate the variable p. Your arrow diagram shows that you can undo the addition of 20 by using the inverse operation, subtraction of 20. If you subtract 20 from the right side of the equation, you must subtract 20 from the left side in order to keep the equation "balanced."

$30 = p + 20$	Original equation
$30 - 20 = p + 20 - 20$	Subtract 20 from both sides of the equation
$10 = p$	Combine $30 - 20$ to get 10 and $p + 20 - 20$ to get p

The solution is 10. When you replace p with 10 in the original equation, $30 = 10 + 20$ is a true statement.

   c) Represent a shift (–12) cipher with an equation.
   d) Solve the equation from (c) when the coded value is 4. Explain how you used inverse operations.

2. For (a)–(f), solve for p.
   a) $43 = p - 4$
   b) $14 = p + 2$
   c) $5 + p = 2$
   d) $5p = 35$
   e) $p - 7 = 15$
   f) $36 = 3p$

3. a) When you decode the value 8 that was coded with the process represented by $c = p - 12$, what equation do you solve? What is the solution?

   b) How is your solution related to the graph of $c = p - 12$?

4. Are the pairs (5, 25) and (23, 3) solutions to the equation $c = p + 20$?

5. When you decode several coded values, it is quicker to use the decoding equation than to solve a separate equation for each coded value. For example, if the coding process is $c = p + 5$, then the decoding process is $p = c - 5$. To decode a coded value like 13, you can replace c with 13 in the decoding equation: $p = 13 - 5$.

   Write a decoding equation for each coding equation. (Hint: An arrow diagram may help.)

   a) $c = 2p$

   b) $c = p - 4$

   c) $c = p + 10$

   d) $c = 5p$

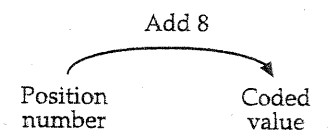


Figure 1.41. A coding process.

Plaintext letter	Coded value
A	2
B	4
C	6
D	8
E	10

Figure 1.42. Part of a coding process.

6. Decode each message.

   a) The arrow diagram in **Figure 1.41** represents the coding process for this message:

   19 22 23 31 17 22 15 16 23 31 28 23.

   b) The equation $c = p - 2$ represents the coding process for this message:

   0 19 7 10 2 17 3 1 16 3 18
   11 3 17 17 –1 5 3 17.

   c) The table in **Figure 1.42** represents the coding process for this message:

   46 18 24 24 16 10 24 32 50 30 42.

   d) The graph in **Figure 1.43** represents part of the coding process for this message:

   25 6 16 10 25 13 10 18 6 21 6 23 25.

   e) Combine the messages from parts (a)–(d).

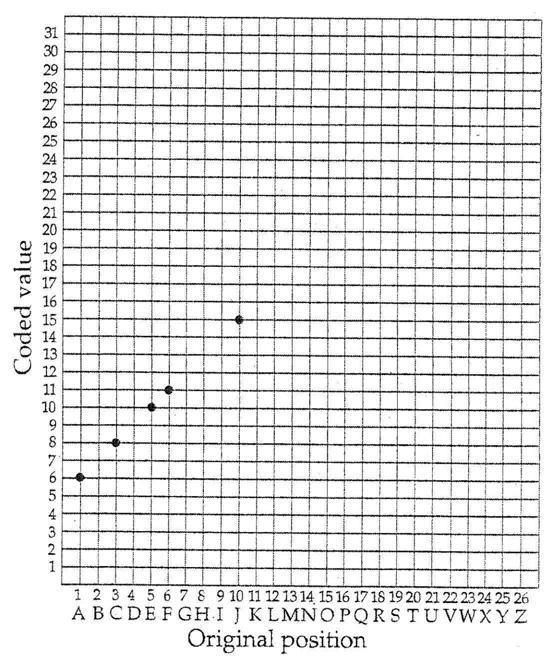


Figure 1.43. Part of a coding process.

7. a) **Figure 1.44** shows four representations of the same coding process. Decode this message:

4 15 24 13 5 16 15 2 19 2 24 9 13 24 17 17 2 15 11 16.

GRAPH

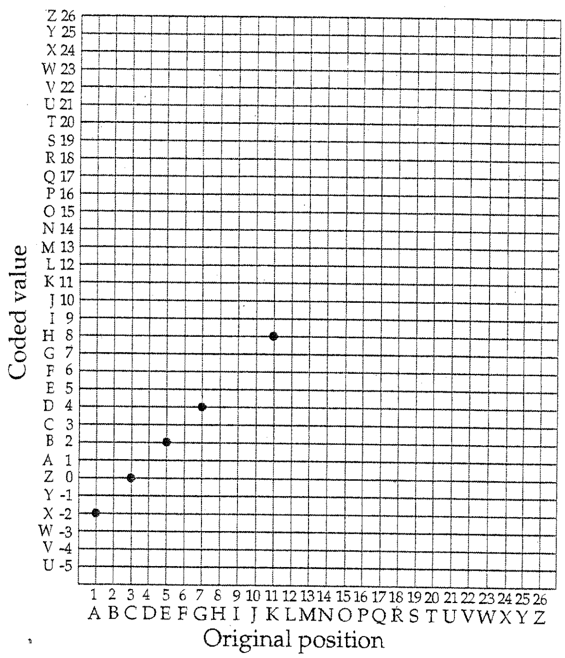
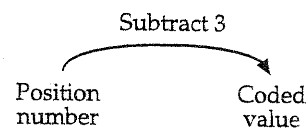


Figure 1.44.
Four representations of the same coding process.

TABLE

Plaintext letter	Coded value
A	−2
B	−1
C	0
D	1
E	2

ARROW DIAGRAM



EQUATION

$c = p - 3$

b) Describe this coding process in words.

c) Which representation did you use to decode? Explain why you chose this one over the other three.

d) What is the domain for this coding process?

e) What is the range for this coding process?

8. a) A decoding process is shown in **Figure 1.45.** Diagram the coding process.

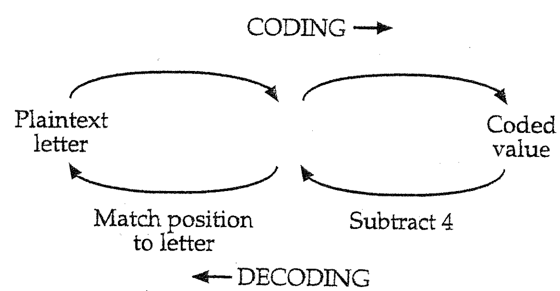


Figure 1.45.
A decoding process.

b) Describe the coding process in words.

9. The graph in **Figure 1.46** represents a shift cipher.

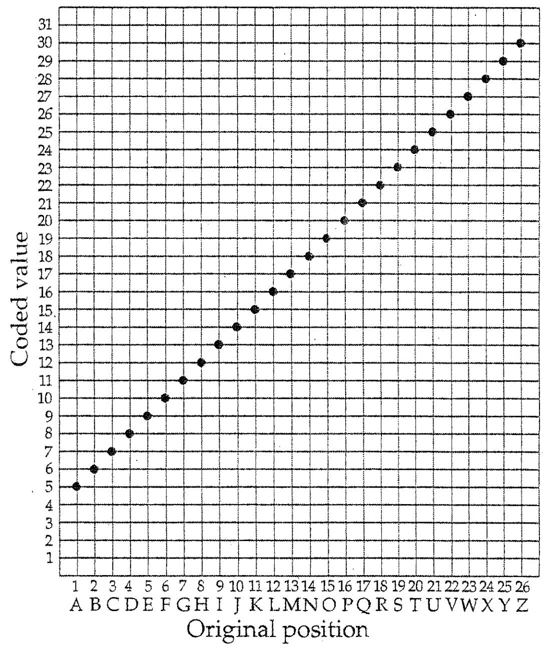


Figure 1.46.
A graph of a shift cipher.

a) Explain why this coding process is a linear function.

b) What is the meaning of the point (6, 10) in the coding context?

c) Decode the number 16.

d) What equation represents the decoding process?

10. Suppose you graph $c = p + 5$ as $Y_1 = X + 5$ on a graphing calculator.

a) Is (3, 8) a point on the graph?

b) Is (3, 8) a solution to the equation $c = p + 5$?

c) In the context of coding, what is the meaning of the point (3, 8)?

d) Can a point not on a line be a solution to the equation used to create the line?

11. a) The word secret was coded as M Y W L Y N. Write symbolic equations for the coding process and the decoding process.

b) Describe how you found your answer in part (a).

12. Suppose you code using two steps, a shift (+8) followed by a shift (+3). Describe or diagram how to decode.

13. Are shift and stretch ciphers easy to decode? Explain.

Activity 1.5: Combination Codes

In this activity, you create a coding model that combines two mathematical operations. You explore two-step coding and decoding processes and examine their effectiveness.

Shift and stretch ciphers are easy to encode and decode. But they are also easy to crack because the coded messages have patterns that are easy to detect.

Suppose you add another step. You might wonder if a process that combines multiplication (stretch) and addition (shift) is better. Use what you've learned about arrow diagrams, equations, graphs, and tables to decide if a two-step coding process is effective.

> A **combination code** combines a stretch and a shift in either order.

1. Code the following message using a process that stretches 2, then shifts +3.

   The FBI says to be silent.

2. The following message was coded with the same process. Decode it.

   21 29 45 41 43 7 13 37 45 21 13 43

3. Is this coding model easy to code and decode?

In Questions 4 and 5, you decide if a stretch followed by a shift gives the same results as a shift followed by a stretch. Also, you decide if two different coding processes can give equivalent results.

4. Consider two coding methods:

   Method 1 stretches 2, then shifts +1 (see **Figure 1.47**).

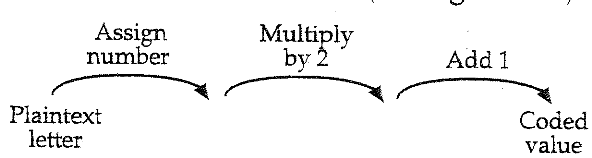


Figure 1.47. A two-step coding process.

   Method 2 shifts +1 first, then stretches 2 (see Figure **1.48**).

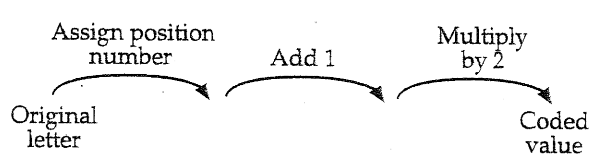


Figure 1.48. Process in Figure 1.47 in reverse order.

   Does changing the order give the same or different results? Justify your answer.

5. a) Use a graphing calculator to make a table for the process in **Figure 1.49.**

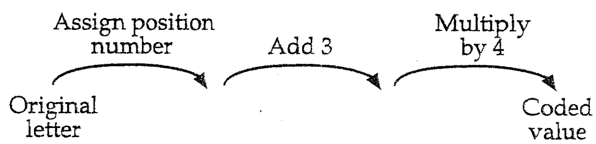


Figure 1.49.
A coding process.

b) Use a graphing calculator to make a table for the process in **Figure 1.50.**

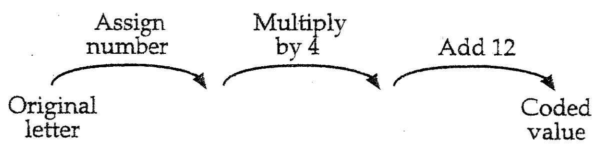


Figure 1.50.
A coding process.

c) Do the two coding processes give equivalent results? Explain.

Activity Summary

In this activity, you:

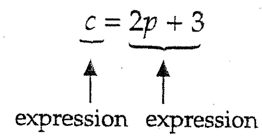
- used two-step coding processes (combination codes) to code and decode.
- explored if changing the order of stretching and shifting gives different results.
- explored if it is possible for two different processes to give equivalent results.

Individual Work 1.5: Stretching and Shifting

In this Individual Work, you review order of operations and the distributive property. You also explore what happens to the graphs of two-step coding processes when you change the control numbers.

TAKE NOTE In an equation such as $c = 2p + 3$, the equal sign is not a variable, constant, or operation symbol. So equations differ from expressions. Equations can be solved; expressions can be evaluated.

An **algebraic expression** is a collection of variables, constants, and operations. $7y$, $3x + 2$, and $4(n - m)$ are examples. These expressions are the building blocks of equations.



In Questions 1–11, you examine rules for algebraic expressions.

Two expressions are **equivalent** if they always give the same result. Two coding processes are equivalent if they code all letters the same way.

Consider a process that changes a plaintext letter to a position number followed by two mathematical operations: add 4 to the position number and then multiply by 3 (see **Figure 1.51**).

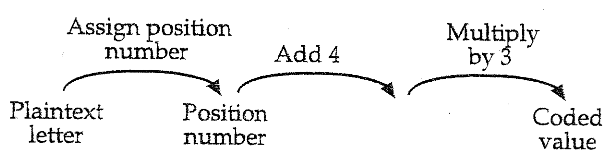


Figure 1.51. A two-step coding process.

This process can be represented by the equation $c = 3(p + 4)$.

Notice the parentheses around $p + 4$. Parentheses ensure that addition is done before multiplication. Without parentheses, the coder would see the equation as $c = 3p + 4$ and multiply first.

In Activity 5, you found that the order in which you do the operations can matter. Operations in an expression are done in the following order:

a) Evaluate expressions within symbols of inclusion such as parentheses.

b) Simplify powers (apply exponents).

c) Do multiplication and division as they occur from left to right.

d) Do addition and subtraction as they occur from left to right.

Keep in mind these rules as you work with equations and expressions.

1. In (a)–(d), evaluate each expression.

   a) $7 - 3 + 9$

   b) $10 \div 5 + 3 \cdot 4$

   c) $4 - (3 \cdot 2)^2 + 5$

   d) $2 \cdot (12 - 8) + 1$

2. For (a)–(c), write a symbolic equation for each coding process. (Hint: Don't forget to use parentheses when needed.)

   a) Add 6 to a letter's position number, then multiply by 9.

   b) Multiply a letter's position number by 2, then subtract 5.

   c) A stretch of +5, followed by a shift of –4.

3. For (a)–(c), describe the coding process that each equation represents.

   a) $c = 4p + 1$

   b) $c = 4(p + 1)$

   c) $c = 4p - 1$

In Questions 4–12, you see that order of operations is also important to the decoding process. The arrow diagram in **Figure 1.52** shows how to reverse the operations in the decoding process.

A symbolic equation for decoding is $p = \frac{c}{3} - 4$ or $p = c/3 - 4$. Parentheses are not needed because division is done first.

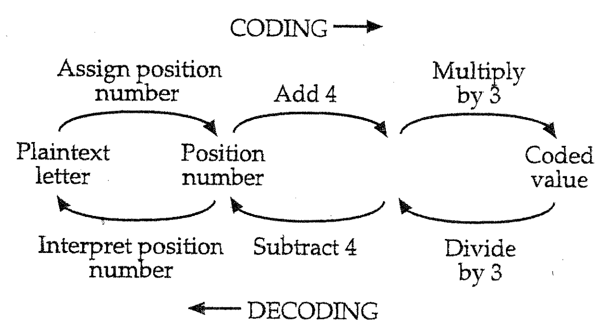


Figure 1.52.
An arrow diagram for coding and decoding.

Suppose you code a position number by multiplying by 2, then subtracting 3 (see **Figure 1.53**).

The coding equation is $c = 2p - 3$. The decoding equation is $p = \frac{c+3}{2}$ or $p = (c + 3)/2$.

In the first decoding equation, the fraction bar acts as a symbol of inclusion. In the second equation, parentheses do the same thing. The fraction bar and parentheses ensure that addition happens before division.

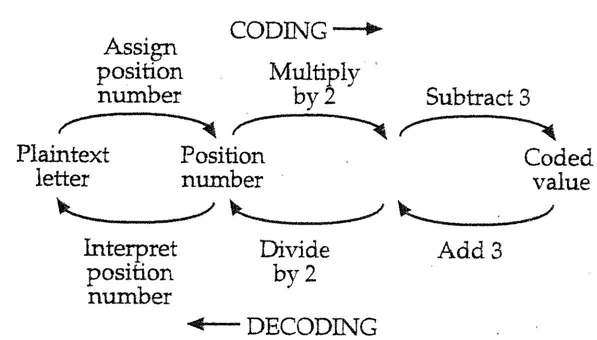


Figure 1.53.
Coding and decoding processes.

4. The process used to code a message is "Add 1 to a letter's position number, then multiply by 2." This process shifts before stretching.

   a) Draw an arrow diagram for the coding process.

   b) Write a symbolic equation for the coding process.

   c) Add arrows to your arrow diagram from part (a) that show the operations in the decoding process.

   d) Write a symbolic equation for the decoding process.

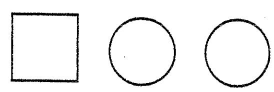
5. a) Draw an arrow diagram for a coding process with a stretch of 2 followed by a shift of +5.

   b) Add the decoding process to the arrow diagram.

   c) Write an equation for the coding process and an equation for the decoding process.

6. One way to show that two expressions are equivalent is with geometric representations. For example, you can use a square to represent the variable and a circle to represent the constant 1. The expression $x + 2$ is shown in **Figure 1.54.**

Figure 1.54.
A representation for the expression $x + 2$.



   a) Use squares and circles to represent a coding process that stretches 2, then shifts 3. (Hint: first the variable is doubled, then 3 is added.)

   b) Suppose you change the order of the operations in the coding process. Use squares and circles to represent a coding process that first shifts 3, then stretches 2.

   c) Are the processes equivalent? Explain.

7. Use squares and circles to show why the coding processes $c = 2(p + 3)$ and $c = 2p + 6$ are equivalent.

Question 7 is an example of the **distributive property.**

The Distributive Property

The product of a and $(b + c)$ is

$a(b + c) = ab + ac$ or $(b + c)a = ba + ca.$

For example, $2(3p + 7) = (2)(3p) + (2)(7) = 6p + 14.$

8. Use the distributive property or circles and squares to create equivalent expressions with parentheses removed.

   a) $5(p + 5)$

   b) $4(2x + 3)$

   c) $2(x + 7) + 2$

9. The squares and circles in **Figure 1.55** represent a coding process. Describe or diagram a decoding process.

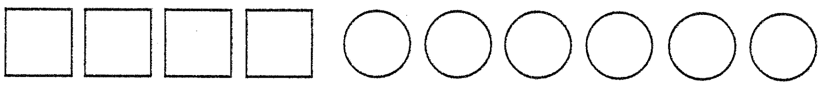


Figure 1.55. Representation of a coding process.

10. Consider the two-step coding process $c = 3p - 7$.

    a) Draw an arrow diagram for both the coding and the decoding process.

    b) Use your arrow diagram to write a decoding equation.

11. Each of these represents a coding process. Describe the steps for decoding or make an arrow diagram for the decoding process.

    a) $c = 4p + 1$ b) $c = 2(p + 3)$

    c) $c = 3p - 4$ d) $c = 5(p + 2)$

In Question 11, you used words or an arrow diagram to represent the decoding procedure. In Question 12, you do the decoding process symbolically by solving a two-step coding equation.

One example:

$53 = 3p + 5$	Original equation
$53 - 5 = 3p + 5 - 5$	Subtract 5 from both sides of the equation
$48 = 3p$	Combine $53 - 5$ to get 48 and $3p + 5 - 5$ to get $3p$
$\frac{48}{3} = \frac{3p}{3}$	Divide both sides of the equation by 3
$16 = p$	Divide 48 by 3 to get 16 and $3p$ by 3 to get p

Another example:

$3(x + 1) - 4 = 17$	Original equation
$3x + 3 - 4 = 17$	Use the distributive property
$3x - 1 = 17$	Combine $3 - 4$ to get -1
$3x - 1 + 1 = 17 + 1$	Add 1 to both sides of the equation
$3x = 18$	Combine $3x - 1 + 1$ to get $3x$ and $17 + 1$ to get 18
$\frac{3x}{3} = \frac{18}{3}$	Divide both sides of the equation by 3
$x = 6$	Divide $3x$ by 3 to get x and 18 by 3 to get 6

12. Use symbolic procedures to solve each equation.

    a) $36 = 2x - 10$

    b) $53 = 5x + 8$

    c) $2(x + 7) = 40$

    d) $3(x - 6) = 36$

13. Suppose $m = 5(n + 2) - 3$.

    a) What is the value of m when $n = 8$?

    b) Find the value of n when $m = 67$. What equation do you solve?

14. The following coding processes use division. Are they equivalent? Explain.

    Add 6, then divide by 2.

    Divide by 2, then add 6.

15. You can use a graphing calculator to solve the equation $2x + 13 = 27$. Let $Y_1 = 2X + 13$ and $Y_2 = 27$. Find the coordinates of the point at which the two lines intersect. The x-coordinate is the solution.

For Questions 16–19, recall that in the coding process $c = p + 3$, p and c are variables. 3 is a constant. It acts as a control number. Changing the constant gives a different equation with a parallel graph.

In a coding process with two steps such as $c = 2p + 3$, 2 and 3 are constants. Both act as control numbers. In Questions 16–19, you investigate what happens to the graph when you change one or both of the control numbers.

16. The equation $c = p + 3$ represents a shift of 3 and no stretch. The equation $c = 2p + 3$ represents a stretch of 2 followed by a shift of 3.

    a) Use the graphing calculator to graph both $c = p + 3$ and $c = 2p + 3$. Describe how the graphs differ.

    b) Predict how the graph of $c = 3p + 3$ differs from the graphs of the two equations in (a). (Write your prediction then use the graphing calculator to check it.)

    c) How can you tell from a graph if a stretch was used in the coding process?

TAKE NOTE The graphs that you see on your calculator are continuous lines that represent all ordered pairs that make the equations true. Remember that the graphs of the coding processes are a series of discrete points that represent only those ordered pairs that make sense in a coding context.

17. Consider the coding processes described by $c = 3p + 2$ and $c = 3p + 5$.

    a) Without graphing, predict how the graphs are similar and how they differ. Test your predictions by graphing.

    b) Do these processes code any letters the same? How do you know?

18. Compare the coding processes $c = 3p + 2$ and $c = 2p + 2$.

   a) Predict how the graphs are similar. Graph both lines to confirm your predictions.

   b) Do these processes code any letters the same? How do you know?

   c) Do $c = 3p + 2$ and $c = 2p + 2$ represent linear functions? Explain.

19. a) Suppose you code using the process $c = -2p + 4$. What is the change in the coded value when you move from one plaintext letter to the next?

   b) Predict how the graph of $c = -2p + 4$ differs from the graph of $c = 2p + 4$. Use a graphing calculator to check your prediction.

20. A coding process is a function that matches or maps a plaintext letter to another letter, number, or character. There are many other processes that map or match a number from one set with one number from another set.

   Converting one measurement of length to another is such a function. The ruler in **Figure 1.56** measures length in centimeters on one edge and inches on the other. One inch is matched to 2.54 centimeters.

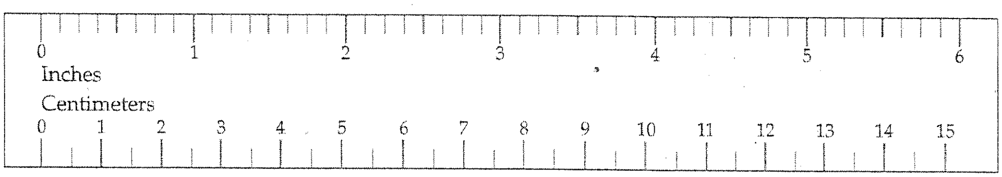


Figure 1.56.
A ruler with centimeter and inch scales.

   a) How is 3 cm *coded* as inches?

   b) How is 8 inches *coded* as centimeters?

21. A calculator with a list feature can be used to make a coding table for $c = 4p + 5$. First you must store the original position numbers {1, 2, 3, . . . , 26} as a list. Do the coding process in two steps by multiplying the list by 4 then adding 5 (see **Figure 1.57**).

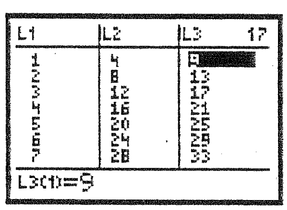


Figure 1.57.
Coding with calculator lists.

   Use the lists to find the coded values for the letters in *matrix*.

22. In Individual Work 1.3 you found that you can add two matrices if they have the same dimensions. It is possible to multiply a matrix by a given number. To do so, multiply each member by the number. For example,

$$4\begin{bmatrix} 7 & -3 & 6 \\ 0 & -5 & 1 \\ 4 & 11 & -8 \end{bmatrix} = \begin{bmatrix} 4(7) & 4(-3) & 4(6) \\ 4(0) & 4(-5) & 4(1) \\ 4(4) & 4(11) & 4(-8) \end{bmatrix} = \begin{bmatrix} 28 & -12 & 24 \\ 0 & -20 & 4 \\ 16 & 44 & -32 \end{bmatrix}.$$

For (a) and (b), use the following matrices:

$$M = \begin{bmatrix} -5 & 8 \\ 2 & -10 \end{bmatrix}, N = [-8 \quad 12 \quad 34 \quad -2].$$

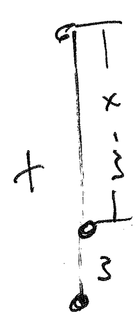
a) Calculate $5M$.

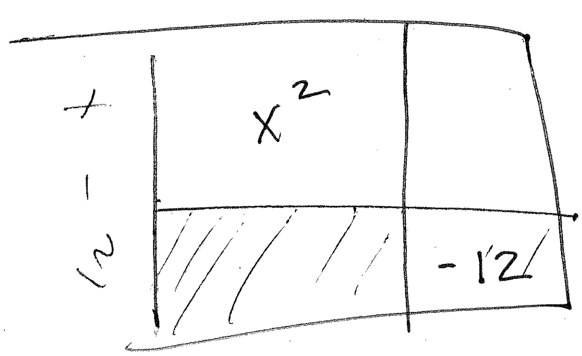
b) Calculate $-1N$.

c) In Individual Work 1.3 you coded a message using matrices and a shift cipher. In a similar way, you can code messages using matrices and combination codes.

Use the steps below to code the word *matrix* with the coding process $c = 4p + 5$.

- Change *matrix* to position numbers.
- Enter the position numbers in the calculator as a 1×6 or 2×3 matrix. Name this matrix [A].
- Make a matrix [B] with the same dimensions as [A] and that contains all 5s.
- Do the matrix operations 4[A] + [B], and write the coded values.





Activity 1.6: Crack the Code

In previous lessons, you learned how to code and decode messages. In this activity you crack coded messages by looking for patterns. You also examine the effectiveness of the coding processes.

> **TAKE NOTE**
> As you try to crack the code, remember the kinds of coding processes you have studied:
> - Shift ciphers
> - Stretch ciphers
> - Combination (two-step) codes

1. Crack the message below. Use anything you have learned about coding and decoding. Look for patterns that might give away the coding process.

   21 9 28 16 13 21 9 28 17 11 17 9 22 27 16 9 30 13 14
   23 29 22 12 9 31 9 33 28 23 11 23 12 13 21 13 27 27 9
   15 13 27 28 16 9 28 11 9 22 22 23 28 10 13 12 13 11 23
   12 13 12 13 30 13 22 17 14 33 23 29 19 22 23 31 28 16
   13 11 23 12 17 22 15 24 26 23 11 13 27 27

2. Explain the methods you used to crack the code. Or, if you didn't crack it, explain how you tried to.

   If you didn't crack the code, perhaps you could use more information. If you did crack the code, this part of the activity will help you crack other codes.

3. Code breakers use statistics. They rely on the fact that some letters are used more often than others.

   a) Which letters do you think appear most in the English language?

   b) Which letters do you think appear least in the English language?

   Look at the table in **Figure 1.58.** It shows a **frequency distribution** for letters of the alphabet.

E	T	N	R	I	O	A	S	D	H	L	C	F
0.130	0.093	0.078	0.077	0.074	0.073	0.073	0.063	0.044	0.035	0.035	0.030	0.028
P	U	M	Y	G	W	V	B	X	K	Q	J	Z
0.027	0.027	0.025	0.019	0.016	0.016	0.013	0.009	0.005	0.003	0.003	0.002	0.001

Figure 1.58.

The letters are listed by frequency. The number below a letter is the fraction of the time that it appears in the English language. The most common letter, E, is listed first, and the least common, Z, is listed last.

4. Here is a message coded with a combination code. Use Figure 1.58 to help you crack it.

   5 59 62 56 5 44 23 17 14 47 11 65 41 17 44 62
   11 5 38 38 17 14

   62 26 17 68 47 77 44 29 11 26
   41 5 44 65 59 11 56 29 50 62

   26 5 59 20 47 29 38 17 14 11 47 14 17
   11 56 5 11 35 17 56 59

   59 29 44 11 17 29 62 59 14 29 59 11 47 68 17 56 77
   47 68 17 56

   62 26 56 17 17 26 65 44 14 56 17 14 77 17 5 56 59
   5 23 47

5. Is a shift cipher easy to crack? Is a shift cipher a good coding process? Explain.

6. Are combination codes easy to crack? How do they compare to shift ciphers?

7. Can you think of another coding process that might be better than shift ciphers or combination codes?

Activity Summary

In this activity, you:

- used frequencies of occurrence of letters to crack coded messages.
- examined the effectiveness of coding processes.

Individual Work 1.6:
Code Cracking

In this Individual Work, you use your knowledge of coding and decoding, and linear functions to crack coded messages. You also use what you have learned about letter frequencies.

1. This message was encoded with a shift cipher. Crack the code and decipher the message. Explain your strategy and any assumptions you make.

   25 30 17 19 25 32 24 21 34 35 41 35 36 21 29 21 17 19 24 35 41 29 18 31
   28 31 22 36 24 21 29 21 35 35 17 23 21 39 25 28 28 18 21 34 21 32 34 21
   35 21 30 36 21 20 18 41 17 30 31 36 24 21 34 35 41 29 18 31 28 34 17 36
   24 21 34 36 24 17 30 18 41 18 28 31 19 27 35 31 22 35 41 29 18 31 28 35
   17 35 39 31 37 28 20 18 21 36 24 21 19 17 35 21 22 31 34 17 19 31 20 21

2. Decipher the following message. Explain your strategy and assumptions.

   G R Z N U A M N S G Z N K S G Z O I O G T Y N G B K H K K T O
   T B U R B K J O T I X E V Z U G T G R E Y O Y L U X G R U T M Z
   O S K O Z C G Y U T R E O T Z N K Z·C K T Z O K Z N I K T Z A X
   E Z N G Z S G Z N K S G Z O I Y N G Y H K K T V A Z Z U C U X Q
   Y E Y Z K S G Z O I G R R E O T Z N K J K Y O M T U L I U J K Y G
   T J I O V N K X Y

3. a) Crack the code. Decipher the coded portion of the quote below.

   "Mathematicians have recently devised a system of coding messages that allows you to tell everyone how to code a message without worrying that it will be deciphered by someone for whom it wasn't intended.

   60 24 27 57 57 75 57 60 15 39 33 42 45 69 42 3 57 48 63 6 36 27 9 33
   15 75 9 54 75 48 60 45 21 54 3 48 24 75 3 48 48 36 27 15 57 3 6 54 3
   42 9 24 45 18 39 3 60 24 15 39 3 60 27 9 57 9 3 36 36 15 12 42 63 39 6
   15 54 60 24 15 45 54 75 to allow governments, banks, and others to receive coded messages from anyone."

   b) Is this a shift cipher? How do you know?

4. What is "decor"? An answer to this question is shown below in coded form. The coding process used was "Add 1 to a letter's position, then multiply by 2." To make cracking the message harder, the coder did not show the blanks between the words.

   42 18 12 34 4 38 42 32 14 42 18 12 4 34 34 26 12 52 32 44
   42 18 38 32 48 4 48 4 52

   a) Decode the message.

   b) What do the coded values have in common?

   c) Explain why this occurs.

   d) Describe other codes with this characteristic.

   e) Give an example of a coding process for which all the coded values are odd.

5. You get a message that was coded using $c = 2(p + 1)$, and one of the characters is a 7. What can you conclude?

6. a) You are trying to crack a coded message and you suspect that A, B, and C are coded as shown in **Figure 1.59.** How do you think the letter D is coded?

   b) Write an equation for the coding process.

   c) Write an equation for the decoding process.

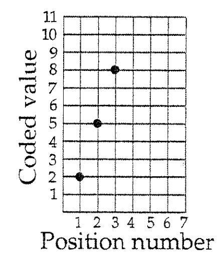


Figure 1.59.
Part of a coding process.

7. Once you see a pattern in the graph of a coding process, you can write a symbolic equation for the process.

   a) How do you find an equation of a linear function from its graph?

   b) Describe the coding process represented by the graph in **Figure 1.60.**

   c) What is the equation for the coding process?

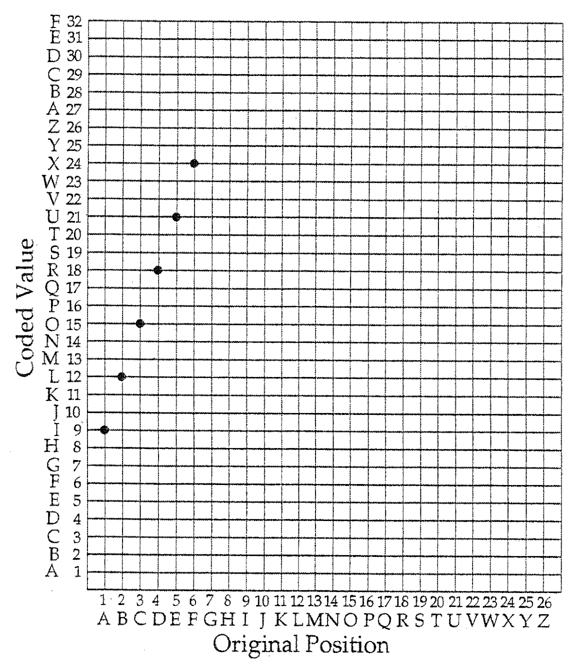


Figure 1.60.

8. Everyone in a company is given a weekly raise of $20.00. An arrow diagram can be used to show this process (see **Figure 1.61**).

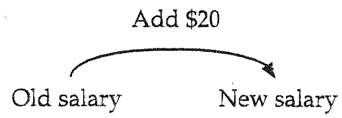


Figure 1.61.
Arrow diagram for a salary raise of $20.00.

a) Use the arrow diagram to write a word equation. Then use n for new salary and s for old salary and write a symbolic equation.

b) If your new salary is $194, what equation can you solve to find the old salary?

c) The company gives employees the option of taking a 5% raise followed by a $10 increase. Draw an arrow diagram to represent a 5% raise followed by a $10 increase.

d) If you were an employee for this company which option would you choose?

Option A: $20 increase

Option B: 5% raise followed by $10 increase

Explain your choice.

9. In *The Adventure of the Dancing Men* by Arthur Conan Doyle, Sherlock Holmes believes the drawings of dancing men are more than just sketches. He unlocks the secret of the dancing men and astounds everyone. Perhaps you can decipher the secret characters as shown in **Figure 1.62.** (In the story, Abe Slaney writes the messages to Elsie, the wife of Mr. Hilton Cubitt.)

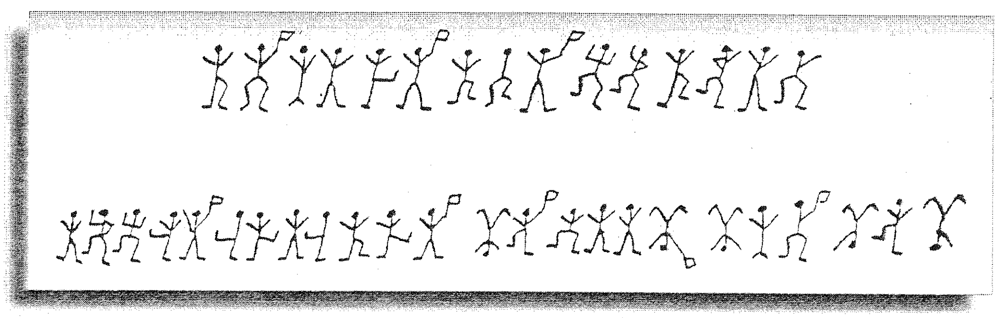


Figure 1.62.
The dancing men coded message.

10. The preparation reading for Lesson One has a coded message. You now have the tools you need to crack the code and decipher it. Do so.

I Y P K P I I P C F P I Q C B T D F D C P I Y P N T F I J T N N T Q K
P I I P C F B Q I Y P P Q V K B F Y K D Q V L D V P

Research Activity: Letter Pairs

In this lesson, you examined how often various letters occur in the English language. Code crackers also look for patterns in letter pairs. For example, in the words *crackers*, *pattern*, and *letter* the combination *er* appears.

- Which letter pairs do you think occur most often in the English language? Make a prediction and write it down.
- Find a page in a book you are reading. Do a frequency study of letter pairs.
- Do another frequency study on a different page. Compare the results to those you found for the first page.

Write a summary of what you discovered. Discuss how a code breaker could use your findings to help crack coded messages.

If appropriate, display your results in a bar graph.

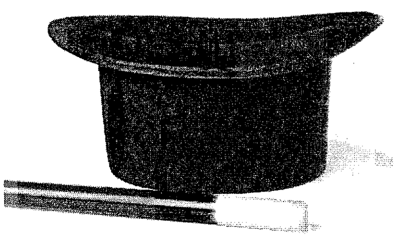
Activity 1.7: Number Tricks

In this activity, you decide if a multistep coding process is more secure than a two-step one.

A magician's number tricks are similar to coding processes. A number trick has many steps and is designed to confuse the audience. Perhaps a coding process with many steps will confuse the code cracker.

One of the questions you seek to answer is how the magician is able to decode the final number quickly. Use your mathematical tools to find out if patterns exist that are useful to the magician.

Here is a sample number trick.



Choose a number.

Multiply the number by 3.

Add 4.

Multiply the result by 2.

Add 5.

Report your result.

1. a) Make an arrow diagram for the magic trick. Begin with "Choose a number" and end with "Final number."

   b) Use mathematical tools to find patterns or clues used by the magician.

   c) Use the results of your search to describe how a magician can find the original number quickly.

2. a) Pretend the magic number trick in Question 1 is a coding process and code the word *magic* as numbers.

   b) Decode the number 43.

3. Consider this number trick.

   Pick a number from 1 to 10.

   Add 1.

   Add your starting number.

   Subtract 6.

   a) Make a table that shows the results for the starting numbers 3, 4, 7, and 8.

   b) Make a graph using the pairs in your table.

c) Your four points should lie on a straight line. Use a straightedge to draw the line. The graph should only show values from 1 to 10 on the horizontal axis because the number trick is limited to numbers from 1 to 10.

d) Someone tells the magician that the ending number is 7. Use your graph to find the starting number.

e) Someone tells the magician that the ending number is 2. Find this on your graph. What is the starting number? Is this possible?

f) Suppose your graph represents a coding process rather than a number trick. Decode the numbers 7 and 2. Do these values make sense in the coding context?

4. The previous questions focused on coding and decoding a multistep process disguised as a number trick. This message was coded using a multistep coding process. Check to see if it is harder to crack.

45 19 19 17 47 39 33 39 35 39 37 15 39 17 19 47 11 45 19 51
47 19 17 13 59 15 39 35 41 11 37 27 19 47 33 27 31 19 47 39
37 59 49 39 19 37 47 51 45 19 49 25 19 21 27 17 19 33 27 49
59 39 21 35 51 47 27 15 55 25 19 37 27 49 27 47 41 33 11 59
19 17 13 11 15 31 21 45 39 35 15 39 35 41 11 15 49 17 27 47
31 47

a) Try to crack the code and decipher the message. Use tables and graphs when needed.

b) Describe how you cracked the code. Or, if you were unable to crack the code, describe the methods you tried.

5. Is a multistep coding model a good one? Explain.

Activity Summary

In this activity, you:

- explored number tricks.
- investigated the effectiveness of multistep coding processes.

Individual Work 1.7: The Magic of Algebra

In this Individual Work, you use symbolic methods to show that two expressions are equivalent.

Multistep coding processes are no harder to crack than two-step coding processes. In fact, a multistep process can be thought of as a two-step process in disguise. The disguise works well for the magician but not for the coder.

There are many ways to show that a multistep process is equivalent to a two-step process. One way is to write it symbolically. For example, consider the expression $3(2x + 8) - 12$.

$3(2x + 8) - 12$	Original expression
$3(2x) + 3(8) - 12$	Use the distributive property
$6x + 24 - 12$	Multiply $3(2x)$ to get $6x$ and $3(8)$ to get 24
$6x + 12$	Combine $24 - 12$ to get 12

So the expressions $3(2x + 8) - 12$ and $6x + 12$ are equivalent.

1. Consider this number trick:

   Pick a number.

   Multiply by 2.

   Add 8.

   Multiply by 3.

   Subtract 14.

   a) Start by using x for the starting number. Follow the steps of the magician to create an expression for the result.

   b) Use the distributive property to remove the parentheses and find an equivalent expression.

2. Use the distributive property to change each multistep coding process into a two-step coding process.

   a) $c = 3(5p + 4) - 6$

   b) $c = 2(2p + 12) + 15$

   c) $c = 5(p + 3) - 9$

   d) $c = 8(10p - 3) - 12$

3. Consider this number trick:

   Pick a number.

   Multiply by 4.

   Subtract 2.

   Multiply by 3.

   Add 10.

   a) Draw an arrow diagram for this trick.

   b) Write a symbolic equation for the steps of this trick.

   c) Suppose the final number is 64. Find the original number. What equation do you solve?

4. Here is a number trick:

   Pick a number.

   Multiply by 2.

   Add 6.

   Divide by 2.

   Subtract your original number.

   a) Try this trick with three different numbers. Record the results in a table.

   b) Write an equation for this trick.

   c) How does this trick differ from others you have seen?

   d) Is this number trick process a function? Explain.

5. Show that the equation $c = 3(p + 1) + 6$ is equivalent to the equation $c = 3p + 9$.

6. Consider this number trick:

   Pick a number.

   Double it.

   Add 1.

   Double the result.

   Add 1.

   Double the result.

   Add 2.

   Divide by 8.

   a) Model the trick with an equation.

   b) What would the magician do to find the original number?

7. A number trick or coding process is described by $y = 3(5x + 4) - 6$. Write an equation for the decoding process.

8. a) Find a two-step coding process to match the graph **in Figure 1.63.**

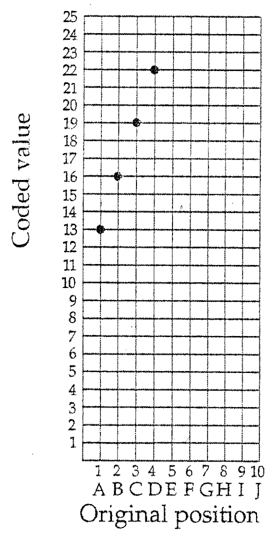


Figure 1.63.
Part of a coding process.

b) Find a multistep (three or more) coding process to match the same graph.

9. Write a multistep process that is equivalent to the given two-step process.

a) $c = 2p + 5$

b) $c = 5p - 15$

c) $y = 0.5x + 1$

10. Can you design a multistep process that makes the resulting number identical to the beginning number? If yes, give an example. If no, explain why not.

11. Write a multistep coding process that codes the word *computer* as

14 50 44 53 68 65 20 59.

12. a) Use a squaring process to code the word *square* as numbers.

b) The word 9 225 16 25 was coded using a squaring process. Decode it.