# What would it look like if we designed schools to be places where teachers learned, alongside their students? 99 <br> —Dr. Elham Kazemi 

## Reflective Teaching

## A Focus on Student Thinking in Problem-Solving

If there is one mantra that has stuck with us when it comes to improving our math instruction, it is "make a small change, reflect, and do it again." In their article, "Math Tasks as a Framework for Reflection: From Research to Practice," Mary K. Stein and Margaret Schwan Smith cite the NCTM Professional Standards for Teaching Mathematics which argue that a primary factor in the professional growth of teachers is the opportunity teachers have to "reflect on learning and teaching individually and with colleagues." They go on to say that whereas all teachers informally think about what happens in their classrooms, "cultivating a habit of systematic and deliberate reflection may hold the key to improving one's teaching as well as to sustaining lifelong professional development."

But what should teachers reflect on? There is no right answer to that question, but we'd like to share some work we've been doing to support teacher reflection, focusing on student mathematical thinking on nonroutine math problems.

Below you will find three sets of questions focusing on three important aspects of your teaching-planning, student work, and reflection/ revision. The goal of these questions is to help you learn from your experience and from the experience of your students. I recently heard an inspiring question from Dr. Elham Kazemi, professor of mathematics education and associate dean for professional learning at the University of Washington-"What would it look like if we designed schools to be places where teachers learned, alongside their students?" We offer the process detailed below as a beginning. Even if you only have time to answer a few questions from each section, or if you only do this formally once a year, we hope the experience will be rewarding.

You are a scientist looking into learning. The planning phase is your problem-posing and hypothesizing. The teaching is the experiment, and the student work is the data collection and observation. The reflection is the conclusion and may lead to a revised hypothesis and a new "teaching experiment."

One final suggestion...Consider doing this with at least one other teacher. You can do a problem together and then work together on the Planning Questions. Then each of you tries the problem with their students. You can meet again and discuss what you learned from your students' reasoning. These questions can help structure any follow-up conversations.

> Make a small change, reflect, and do it again.

Dr. Kazemi has developed a great observation technique called "teacher time out." To learn more, visit http://www. shadowmathcon.com/elhamkazemi/

## Instructions

1 Find or create an open-ended, challenging math problem that meets the criteria proposed in the introduction to the math section.
(2) Complete the Planning Questions below, before doing the problem with your class.

3 Try the problem with your students.
4 Collect samples of student work.
5 Choose some samples of student work and complete Questions on Student Work.

6 Complete the Reflection/Revision Questions.

## PLANNING QUESTIONS

We start with a quality math problem and try to solve it in as many ways as possible. Once we have had our own problem-solving experience with the problem, we can be explicit about the content/strategies we want students to learn. Then we start to think about how to engage students. We can begin to imagine how students might approach the problem. We can also start to identify potential student struggles and plan for them beforehand. One of our goals is to allow students to experience productive struggle and that requires some preparation. It can be hard to come up with questions to support struggling students and extensions to challenge faster learners if we have to do it in the moment. Certainly, we are always going to have to do some thinking on our feet, but the better prepared we are, the more strategic we can be.
(1) Describe how you solved the problem.
(2) Can you think of any other ways to solve the problem?
(3) Why did you choose this problem? What do you like about it?
(4) Why do you think this is a DOK 3 problem?
(5) What do you want students to get from working on this problem?

6 Identify and describe a few specific challenges you think students will have in solving the problem. Describe how you might help and support the problem-solving efforts of those students without giving too much away.
(7) How could you extend this problem for students who finish?

## QUFSTIONS ON STUDFNT THINKING

Adult education teachers always talk about how much we learn from our students. Many teachers say they learn more from their students than their students learn from them. Teachers are usually referring to all the stories and experiences our students share, or the inspiration we derive from their decisions to come to our classes, balancing complicated lives and responsibilities for a regular date with struggle. But there is another really important way we can learn from our students-focus on their reasoning. We must really delve deeply into student thinking, to understand the individuals in our class, and also to better understand adult education students in general and how they learn. Our students are trying to teach us, if we take the time to listen. If possible, consider doing this phase with another teachers. Math teachers coming together to analyze student thinking can be a very rich activity. Remember when choosing work to analyze, don't focus only on students who got the right answer. You may learn more from student mistakes, or solution methods that are interesting but incomplete.

## Answer the following questions for each sample of student work you choose:

(1) Explain each student's method/thinking.
(2) Why did you choose this sample of student work?

What did you learn from it?
(3) How typical was this student's approach in your class?
(4) Any additional comments?

## REFLFCTIONS/REVISION QUESTIONS

Whatever happens is an opportunity to learn something about your students and how they learn. If something doesn't go well, you can learn a lot about how to do it better next time. And if things do go well, why did they go well and how could they go better. This section is about looking back at your predictions and comparing them to what happened-as you observe and analyze student thinking you'll start to improve your sense of how they will make sense of and productively struggle with future problems. Even if you are not going to be sharing this with other teachers, spend some time with the last question. The teacher you are advising might be you.

> Our students are trying to teach us if we take the time to listen.

1 What did you learn from using this problem with your students (about math, about individual students, about your class, about student thinking in general, etc)?

2 What (if anything) would you do differently if you used this problem again?

3 Comment on how your class (individual students, or as a whole) may have benefited from their work on this problem.
(4) Did students get what you wanted them to get from the problem? How do you know?
(5) What challenges came up for your students that you didn't expect?

6 What strategies/solution methods/questions came out that seemed helpful to students?

7 What advice/message do you have for a teacher who is considering using this problem with their class?

## A Call to Action

To give readers a real sense of how helpful these reflections can be, we are including three sample write-ups, written by Tyler Holzer, a teacher leader at a community-based organization in Brooklyn, NY. If you find Tyler's write-ups helpful, consider writing one yourself, using these questions to guide you. Share those write-ups with your colleagues. Write them with your colleagues. If you are a program manager, consider protecting some time for your staff to work on these questions together. We believe in teacher-led professional development of practice. Too often, we teach in our little pocket of the egg carton, isolated from other teachers. Let us turn our classrooms into laboratories to learn about learning and share what we discover.

## - Multiples of Nine Problem

- The Gold Rush Problem
- The Movie Theater Problem


# Multiples of Nine Problem 

## The Problem:

Find the smallest multiple of 9 that has only even digits. Please show all your work.

## How I Solved It

I knew that none of the two-digit multiples of 9 contained only even numbers. I also knew that any multiples of 9 that were between 100 and 199 wouldn't work, because they all would have a 1—an odd numberas the leading digit. I started working under the assumption that the correct number would be somewhere in the 200 s , so I picked a nice, round number and started from there. I calculated $9 \times 30=270$. Because this had a 7 in it, I knew that it couldn't be the right answer, but I noticed that if I were to add 18 to 270 , I would get 288 . Thus, $9 \times 32=$ 288 was my tentative answer.

I couldn't commit to this answer because there might be a smaller multiple of 9 that was located in the 200s and also had only even digits. So I went back to 270 and began counting down by 18. I counted down by 18 instead of 9 because the correct number has to be an even number times 9 (so that I would have an even product). So the numbers I checked were $9 \times 28=252,9 \times 26=234$, and $9 \times 24=216$. None of these worked, so the correct answer must be 288 .

## Other Ways to Solve This Problem

I could just write out all the multiples of 9 and keep going until I found one that had only even digits. This method feels a little risky because-if I were just counting up by 9 to the next multiple rather than multiplying each time-it would be easy to make a mistake somewhere. Even if I were go through and multiply 9 by several numbers, it's likely that I would miss a number at some point.

Another way to solve this involves knowing the divisibility test for 9 . If the sum of the digits in a number add up to a multiple of 9 , then the number itself is divisible by 9 . The sum of the digits in this problem couldn't be 9 , though, because the sum of even numbers can never be odd. The smallest multiple of 9 with only even digits must be the smallest combination of three even numbers that add up to 18 . It would have to be 288.

Also, when the multiples of 9 are organized into a table, an interesting pattern emerges. By looking at the digit sums and the changes to the ones and tens digit, we see some interesting things.

| Table of the First Forty Multiples of 9 |  |  |  |
| :---: | :---: | :---: | :---: |
| $9 \times 1=9$ | $9 \times 11=99$ | $9 \times 21=189$ | $9 \times 31=279$ |
| $9 \times 2=18$ | $9 \times 12=108$ | $9 \times 22=198$ | $9 \times 32=288$ |
| $9 \times 3=27$ | $9 \times 13=117$ | $9 \times 23=207$ | $9 \times 33=297$ |
| $9 \times 4=36$ | $9 \times 14=126$ | $9 \times 24=216$ | $9 \times 34=306$ |
| $9 \times 5=45$ | $9 \times 15=135$ | $9 \times 25=225$ | $9 \times 35=315$ |
| $9 \times 6=54$ | $9 \times 16=144$ | $9 \times 26=234$ | $9 \times 36=324$ |
| $9 \times 7=63$ | $9 \times 17=153$ | $9 \times 27=243$ | $9 \times 37=333$ |
| $9 \times 8=72$ | $9 \times 18=162$ | $9 \times 28=252$ | $9 \times 38=342$ |
| $9 \times 9=81$ | $9 \times 19=171$ | $9 \times 29=261$ | $9 \times 39=351$ |
| $9 \times 10=90$ | $9 \times 20=180$ | $9 \times 30=270$ | $9 \times 40=360$ |

## Why I Chose This Problem

I recently discovered this problem, and I really like it for a number of reasons. First, it requires a little bit of vocabulary in order to get started. Students will have to know what a multiple is, they will have to know what digits are-and more specifically, how digits can differ from numbers-and they'll have to understand the difference between even and odd numbers. I also like how nonintimidating it looks at first glance. "How hard could it be to find a multiple of 9 that has only even digits? I shouldn't have to count up very far." Because the problem doesn't look lengthy or challenging, it comes as a surprise when the correct answer is actually the 32nd multiple of nine. I anticipate a lot of students writing out $9,18,27,36,45,54$, etc, and then getting frustrated or giving up when they don't get to the answer fairly quickly.

Second, I like that the problem requires students to perform basic calculations and that it requires precision in order to get the answer right without making mistakes. Also, the repetition involved in either adding 9 over and over or multiplying by 9 over and over is helpful for students with lower math abilities, and it still provides good practice for students who are more comfortable working with numbers. Moreover,
the 9 times tables are interesting because of the pattern that arises in the tens digit and the ones digit. My hope is that as students start listing out the multiples of 9 , they will be able to see the pattern and work with it. I like exposing my students to several different ways of thinking about multiplication. My hope is that they find one that works for them.

And finally, I really like this problem because there are good extension questions. If a student finishes early, they can find the next smallest multiple, and then the next one. Once everyone has had plenty of time to work, the class can talk about divisibility tests, and they could work on finding all the three-digit multiples of 9 that have only even digits. And so on.

## Why This Is a DOK 3 Problem

This is a DOK 3 problem because it invites multiple approaches, and even though there is a correct answer, students need to be able to explain why the number they chose is the correct one. They also have to analyze each multiple they come up with to make sure that both digits are even. (I anticipate several students saying that something like 36 or 72 is correct because it is an even number.) Depending on how quickly students finish, they might be asked to investigate the pattern that shows up in the multiples of 9 that only have even digits. Students could then be asked to draw a conclusion about that pattern.

## My Goal for Student Learning

This problem is intended for a class of new students with low math levels, many of whom struggle with multiplication, and it is going to take a while for most of them to finish. My goal is for them to stick with the problem and not get discouraged as the numbers start getting bigger and bigger. I am giving this problem during the first week of class, and my sense is that the students aren't used to struggling with math problems for long periods of time. Another goal is for students to come up with an organized approach to tackling this problem. That is, I would like to see some students create tables or lists rather than simply start multiplying 9 by randomly chosen numbers. Also, because it's so early in the cycle, I would like to see my students feel comfortable talking about their work and the work of their peers.

## Challenges for Students

The first challenge I anticipate involves the vocabulary and phrasing of the problem. Because I will be working on this problem with a group of new students, they have only recently been introduced to multiples and factors. They will likely need a quick refresher. Similarly, I expect to see
students struggle with the idea of "even digits," and we may have to talk about it as a group to make sure that everyone is on the same page before we get started.

I also expect students to have a hard time organizing their work, and I expect to see some mistakes with basic computation as the multiples get higher and higher. This will require me to intervene somewhat to help students spot their mistakes-either with adding or multiplying, depending on their approach. I think that some students will want to give up after they've found the first fifteen or twenty multiples of 9 . They might think that it's a trick question and that there actually aren't any multiples of 9 that have only even digits.

To support students who are struggling with this problem, I will help them identify mathematical mistakes so that they can correct them as they go along. I won't tell them that they've made a mistake though; instead, I'll ask them to talk about how they got from one number to the next so that they can see the mistake for themselves. I think that some students will notice the pattern in multiples of 9 (increasing tens digit, decreasing ones digit), and so I will help them to articulate it and apply it to the work that they are doing. For those students who work all the way through it and don't see the pattern, I will ask them to look over their work and talk to me about the changes they see to the digits. I expect that some students will try to guess-and-check their way through the problem, which could potentially make it take a very long time. I will talk to these students about ways they might be able to organize their guesses so that they don't lose track of the work they've already done.

## Extension Questions

If some students finish early, I would ask them to find the next smallest multiple of 9 with only even digits, and then the next one, and the next one, and so on. It might seem a little tedious at first, but if I support it well, I can help students to understand how the divisibility test for 9 works. This is something that I don't think many, if any, students will know.

All of the multiples of 9 that are less than 1000 and have only even digits are: $468,486,648,666,684,828$, and 882 . Even if a student only got to 468 and 486 , I could start having the conversation with them about how any number with a digit sum equal to a multiple of 9 must itself be a multiple of 9 . Since no combination of even numbers can sum to 9 , they must have to sum to 18 . From there, students can work on finding the other possibilities.

## Student Work

## FIDEL'S APPROACH

工idel is one of the strongest students in this group. He attends every - class session, asks good questions, and works hard on every problem that he encounters. Even this early in the cycle, Fidel's classmates have come to recognize him as one of the leaders in the class, and they often rely on him to help them out when they are struggling. However, Fidel had a really hard time with this problem.

First off, he needed a reminder on the difference between odd and even numbers, and after we talked about it as a group, he wrote them down just to be sure. Then he started working. If you look closely at Fidel's work, you'll see that he started out by writing all of the multiples of 9 , but then he erased them. When I asked why, he explained that when he got above 100 , he noticed that all of the multiples would have a 1 in them and therefore couldn't be correct. This is where he gave up on the list and decided to try guessing and checking. His guesses look a little disorganized, but there is a method to them. He was trying to locate multiples of 9 that were in the 200s. His first guesses were much too big, but he kept making adjustments. He erased most of these, but he left a few and, after a while he found $9 \times 32=288$.

What was interesting about Fidel's work is how he noticed some important qualities about the numbers-namely, that the correct answer would have to start with a $2,4,6$, or 8 -but he didn't come up with a good way of organizing the work that he was doing. Because he guessed and checked, several students finished the problem before him and began working on the extension questions. This was a case where the strongest student in the class struggled the most because the problemsolving strategy he chose may not have been the most appropriate one.

## JEAN MARIE'S APPROACH

0f all the students in the class, Jean Marie probably has most difficulty with math. She performs all basic calculations on her fingers, and she has very little confidence in her ability to grow as a math student. This was the first extended problem that she had done on her own.

From the outset, Jean Marie was frustrated by this problem because she noticed that it had to do with times tables, and she reminded me several times that she doesn't know her nines. You'll even see at the top of the page that she was drawing circles for the first couple multiples of 9 . While everyone else was working on their own, I spent a lot of time sitting with Jean Marie and talking her through the problem. She

started with $9 \times 1=9$ but then couldn't remember $9 \times 2$. So we talked about how she would figure it out. She seemed a little embarrassed when telling me that she would count on her fingers. But when I told her that her method was fine, she went back to work. She counted up to 18, and then counted up another 9 to 27 , and so on. From here, she was able to work on her own, but she tried to give up about every five minutes. It took a lot of encouragement to get Jean Marie through this problem, and she made a lot of mistakes. I made the decision to help her identify her mistakes so that she wouldn't get more frustrated as she got further and realized she had been working with incorrect numbers.

In the end, with a lot of support, Jean Marie did arrive at the correct answer. I liked how well-organized her method was, and I really appreciated her ability to stick with a problem that was so challenging and frustrating to her. In the end, Jean Marie finished before Fidel did! And it was a really important moment for her. She wrestled with a problem that she thought she could never do, and she was successful.

## FELICIANO'S APPROACH

This was the day when I learned that Feliciano is incredibly good with numbers and loves doing math. He did this problem on his second day in class, and since he was the first to finish, I got to talk to him about some of the extension questions that I was hoping to use.

Feliciano started out by listing the multiples that he knew off the top of his head, and then he worked additively from there. This approach was largely typical of what most students did. Each time he arrived at a new multiple of 9 , he added 9 , wrote the next one down, and repeated. By following this pattern, Feliciano got to 288 pretty quickly, so I asked him to find the next multiple of 9 . He kept working additively for a while before figuring out that $9 \times 52$ was equal to 468 . Here, Feliciano stopped and looked a little more closely
at the relationship between 288 and 468. In the middle of the page, he adds 2 to the hundreds digit in 288 and subtracts two from the tens digit, which gives him 468. He repeats the process again to get 648.

At this point, Feliciano and I talked about why this worked. Feliciano couldn't articulate the divisibility test for 9 , but he was working with it intuitively when he found 468 and 648 . After we talked about how the digits needed to add up to 18 , he was able to find all of the other combinations, which are scattered around the page. I'm glad I got the chance to see how this problem worked with a student who was skilled in math. Feliciano was very engaged with the problem, and he enjoyed getting to learn and talk about the divisibility test.

## Final Thoughts

Treally liked the way this problem played out in class. For most of the students, this was only the second problem-solving activity that they had done. Because they were new to struggling with math problems, I hoped that working on this one would encourage persistence and help them to come up with strategies for organization. For the most part, we met those goals. We also took the time to talk about the patterns that appear in multiples of 9 , as well as the divisibility test, which is shown in the board work at the right. Through working on this problem and its extensions, I learned that with enough preparation, there are interesting questions that can be asked about any mathematical idea-even one as basic
 as multiples.

## ■ What I Might Change

I wouldn't change much about how I did this problem. If I do it again early in the cycle, though, I might try reviewing the different problem-solving strategies that we had discussed before doing this problem. That way, students would have to make a more conscious choice between using a table/chart and trying to guess and check. Unfortunately, this time, a handful of students spun their wheels guessing and checking when they could have used a more effective method. Still, I think they benefited somewhat from doing it the "wrong way" before moving on to a better way.

I might also give out hundreds charts to students who really struggle with their times tables. It could help them to get started, and it could also help them to identify a pattern that will help them remember their nines in the future. And lastly, if I do this problem early in a class cycle again, I might ask students to write a reflection of what it was like working on the problem.

## ■ Unexpected Challenges

I gave this problem again in another class-one with a wider range of math levels-and found that it was a little difficult to manage all of the students. Some students finished the problem quickly, while others needed me to sit with them and keep them working, give them feedback on their work, etc. This made it challenging to keep the higher-level students engaged while still supporting the students who needed individual attention.

## ■ Student Takeaways

My students liked this problem, and it fit in well with the work on factors and multiples that we were doing in class earlier in the week. They enjoyed trying out and discussing some of the problemsolving strategies that we had been working on as a class. They also got to hear about different solution methods from their peers, and they had the opportunity to share their frustrations with the problem, as well as the sequence of steps they took to break through that frustration. For one student in particular-Jean Marie-this problem was a major breakthrough. For the first time in class, she stuck with something, got angry at it, settled back down, tried again, failed, tried again, and finally succeeded. She hasn't given up on a problem since. This is a great exercise to do with students who need to learn how to stick with something. It has a very low entry point, but the discussion can go a lot of different ways.

My students were also able to see the importance of pattern recognition in math. Recognizing the patterns for multiples of 9 helped several students write out all of the multiples quickly, rather than adding repeatedly. After we finished this activity, "Look for a pattern" was added to our list of problem-solving strategies, and it has since helped students succeed in other difficult problems.

## Advice for Teachers

This is a good low-entry problem for students who are new to your class, but it could be used at any point in the cycle as a warmup exercise. Teachers should be prepared for students to get frustrated and give up, but they should also be prepared with extra questions for students who breeze through the exercise. The problem works
best if you allow plenty of time for the class as a whole to debrief, especially because students need to see that there is a bigger takeaway from doing the problem than just crunching numbers. And there's a lot of rich territory on which to have that discussion. Talk about organizing information, talk about patterns, talk about divisibility tests, and emphasize key vocabulary. Help your students understand that their struggle was a productive one.

## The Gold Rush Problem

## The Problem:

In 1848, gold was discovered at Sutter's Mill in California. Over the next several years, hundreds of thousands of prospectors traveled westward hoping to make their fortunes mining gold.

A man named Billy Merrell happened to own some of the land where the gold was discovered. Instead of digging the gold himself, he decided to rent plots of land to the prospectors. Billy gave each prospector four wooden stakes and a rope measuring exactly 100 meters in length. Each prospector then had to use the stakes and the rope to mark off a rectangular plot of land.
(1) Assuming that each prospector would like to have the biggest possible plot, what should the dimensions of each plot be? Explain the reasoning behind your answer in a sentence or two.
(2) One prospector noticed an advertisement that Billy had posted on his land. It read: "Join the ropes together! You can get more land if you work together!"
Investigate whether or not this statement is actually true for two or more prospectors who work together and divide the plot equally, still using just four stakes.

Explain your answer in a few sentences.

## How I Solved It

This is an optimization problem. Let $x$ and $y$ be the dimensions of the rectangular plot. Given the constraint of only having 100 meters of rope, the perimeter of my plot would be $2 x+2 y=100$. The area would be $A=x y$. I started by solving the perimeter equation for y so that I could substitute it into the equation for area.

$$
\begin{aligned}
& 2 x+2 y=100 \\
& \frac{-2 x}{2 y}=\frac{-2 x}{2} \\
& y=50-x
\end{aligned}
$$

Substituting this into the equation for area, I have $A=x(50-x)$, or $A=50 x-x 2$. The graph of this equation will be a parabola with a single
critical point, and that critical point will give me the $x$-value that will maximize area. To find that point, I need to know the derivative of $A$ with respect to $x$; this will be the equation for slope of the tangent line to the graph. The critical point I'm looking for will have a tangent line with slope 0 .

The derivative of my area formula is $A^{\prime}=50-2 x$, where $A^{\prime}$ represents the slope of the tangent line at a chosen point. Since I'm trying to find the point where the slope is zero, I substitute 0 for $A^{\prime}$. Now I have $0=50-2 x$. When I solve this for $x$, I get the solution $x=25$ meters. So this is the optimal length, which means that my optimal width is also 25 meters. The shape that will maximize area is a square that is 25 meters by 25 meters.

To answer the second part of the question, I applied the same rationale to a rope that is now 200 meters in length. If the optimal shape is a square, then it would be 50 meters by 50 meters, and it would have an area of 2500 square meters. This means that each prospector would get 1250 square meters of land, which is twice as much as they would have before. So it does make sense to "join the ropes."

## Other Ways to Solve This Problem

The method I outlined above is impractical for teaching, and I only tried it to challenge myself and to see if I could remember how optimization problems worked. So after solving it algebraically, I wanted to examine the relationship between area and perimeter just so that I could see how much the area changed when I made slight modifications to the dimensions. I drew a few different rectangles and ended up at the square that was my final answer from before.


The pattern I noticed when drawing the rectangles out in this orderfrom long and skinny to square-showed me that as a shape becomes closer in form to a square, the area increases.

I also wrote it out in table form, just so that I could have an organized chart showing the areas given by different dimensions. I started the table at 40 by 10 , as shown below, and worked my way up.

| Length | Width | Area |
| :---: | :---: | :---: |
| 40 | 10 | 400 |
| 39 | 11 | 429 |
| 38 | 12 | 456 |
| 37 | 13 | 481 |
| 36 | 14 | 504 |
| 35 | 15 | 525 |
| 34 | 16 | 544 |
| 33 | 17 | 561 |
| 32 | 18 | 576 |
| 31 | 19 | 589 |
| 30 | 20 | 600 |
| 29 | 21 | 609 |
| 28 | 22 | 616 |
| 27 | 23 | 621 |
| 26 | 24 | 624 |
| 25 | 25 | 625 |
| 24 | 26 | 624 |

The table is interesting because it provides the opportunity to see the consecutive difference in area each time that the dimensions are adjusted by 1 meter. I noticed a pattern, which is added in the updated table below:

| Length | Width | Area | Consec. Diff. |
| :---: | :---: | :---: | :---: |
| 40 | 10 | 400 | $+29=429$ |
| 39 | 11 | 429 | $+27=456$ |
| 38 | 12 | 456 | $+25=481$ |
| 37 | 13 | 481 | $+23=504$ |
| 36 | 14 | 504 | $+21=525$ |
| 35 | 15 | 525 | $+19=544$ |
| 34 | 16 | 544 | $+17=561$ |
| 33 | 17 | 561 | $+15=576$ |
| 32 | 18 | 576 | $+13=589$ |
| 31 | 19 | 589 | $+11=600$ |
| 30 | 20 | 600 | $+9=609$ |
| 29 | 21 | 609 | $+7=616$ |
| 28 | 22 | 616 | $+5=621$ |
| 27 | 23 | 621 | $+3=624$ |
| 26 | 24 | 624 | $+1=625$ |
| 25 | 25 | 625 | MAX |
| 24 | 26 | 624 | $-1=624$ |

## Why I Chose This Problem

I chose the gold rush problem because it is similar to a problem that I had used in my class before; the version I used involved maximizing the area of a garden given a limited amount of available fencing to surround the garden. What I liked about the gold rush problem is that it took the garden problem a step further by asking students to analyze what happens to area when perimeter is doubled, tripled, etc, and I like how it invites different strategies for solving. I also like how this problem encourages creative thinking about basic shapes. It invites students to construct several different rectangles, which ideally will lead them to a better understanding of how dimensions relate to perimeter. Doing this also reinforces computation skills, and so it works well in a multilevel classroom. Once students arrive at their answers, there is a lot to talk about. For example, is a square a rectangle? Is a rectangle a square? Assuming that I did share a plot of land with another prospector, what would be the most equitable way to split it? Do we split the land, or do we split the profits? At what point in joining the ropes together does the workload become too much for one prospector? And so on.

And finally, there's a bit of historical context here, and so it fits well into a class that also has a history component. It acts as a good springboard into a discussion about the pre-Civil War period and the waves of westward migration that were occurring around the time.

## Why This Is a DOK 3 Problem

Depending on how the problem is used and what you're asking your students to produce in the end, this could actually be a DOK 4 problem. At DOK 4, students are essentially completing a research project that allows them to draw a conclusion not just about a specific situation, but about a mathematical concept in general. In this case, the students first have to read and understand a situation that might not be immediately clear to them. From here, they have to think creatively about how they might get started. They then perform some basic calculations, but they also have to keep in mind that they're looking to maximize area while keeping perimeter the same. This attention to multiple constraints and geometric concepts requires some higher-level thinking than a standard DOK 1 or 2 question in which students might just be asked to calculate the area of a shape (level 1), or calculate the area of a shape given its perimeter (level 2).

## My Goal for Student Learning

One goal in presenting this problem is that my students will be able to apply the basic concepts of area and perimeter in a setting that is a little different from what they might be used to. I also hope to see that they're able to think creatively about the problem and make adjustments to the shape of their plots in order to see how the shape of the rectangle has a significant effect on its area. In other words, I want them to be able to create a possible plot but then-without my intervention-try drawing other rectangles as a way of checking to see if their answer is correct. Another goal is for students to verbalize the relationship between the shape of a rectangle and its area. What the students should notice is that, as the rectangles become more square-like, the area increases. I also would like to see an organized approach to solving this problem, although I realize that the way students organize their work will differ greatly.

## Challenges for Students

I anticipate a number of students drawing one rectangle and thinking that they've answered the question after they've successfully calculated its area. Moreover, I anticipate some resistance when I prompt them to try drawing other rectangles so that they can compare the areas of each. I also anticipate some issues with understanding the situation. Even though the prompt specifically mentions rectangular plots, I have a feeling that some students will miss this part. They'll understand that they're getting four stakes and a rope, but they won't really know where to go from there. So I might have to intervene a bit just to clarify exactly what the question is asking them today. I also foresee students jumping to a quick conclusion about the second part of the question. That is, I think that some will gloss over the part about joining two ropes together and just assume that because you're sharing with another prospector, you would get less land.

To support students who are struggling, I will first ask them to tell me what is happening in the problem. I would want to make sure that they understand exactly what they're getting from Billy and why they are getting those materials. If they are unable to make a rectangle, I might ask them to draw one, and then I would ask what the length of the rectangle could be. They could then try a few things and check their work. For students who try to stop after drawing one rectangle, I'll ask how they know that the one they drew provides the most land to work with. So after they try one more, I'll ask that they try another. And so on. I have some students in my class who really struggle to do long multiplication, and so I may allow them to use calculators. The goal of this activity is to encourage reasoning about shapes; it's not about
crunching numbers.

## Extension Questions

If some students finish early, I would ask what would happen if three, four, or five people joined their ropes together. How much land would each person get in these cases? And is there a pattern to the increase in land you get by working together with other prospectors? How could you organize the data to see what the pattern might be? Could this be viewed as an input/output table, or a function? If so, what would be the rule of the function? How do you know? How many ropes would you need to join together so that you could get 7500 square meters to work with?

## Student Work

## ELISA AND BELEN'S APPROACH

Belen is one of the brightest students in the class. Elisa struggles and has missed several classes because of her work schedule and other issues. This group had a hard time getting started, but once they figured out a pattern, they were able to make progress. What I like about their representation is how organized it is. They begin with a rectangle that is 30 meters by 20 meters; it has an area of 600 . The next rectangle they drew had dimensions of 28 by 22, with an area of 616 . When I talked to B. and E. about this, they said that they were surprised about what happened to the area. They explained that they noticed how, when they decreased the length and increased the width, the area got bigger.
So they kept doing this until they arrived at the dimensions 26 by 24 , for an area of 624 . This was the greatest area possible, they said.

When I asked why they didn't go a step further and try 25 by 25 , they reasoned that it wasn't allowed: The plot had to be a rectangle, and 25 by 25 would be a square. I was interested in this solution because I predicted that students would get hung up on the square/rectangle issue, and these two were adamant that the plot could not be a square. So we talked about this. Also, notice their reasoning at the bottom. It says, "After a while we figure it out that if you increase the width, then you have to decrease the length in order to have the same perimeter, but bigger Area." I understood what they meant, but we talked about it for a while to get some clarification. Is there a point at which decreasing length and increasing width doesn't increase area anymore? What is that point? Why does it work this way? This group's graphical approach was very typical of what other students tried.


What interested me about this group's approach was that the first rectangle they drew was actually correct. But they didn't know that. So I prompted them to try drawing a few others. Travis was sure that he could find one with a greater area, because he reasoned that as the length value got bigger and bigger, the area would too. He wasn't really thinking multiplicatively yet. So he tried some other rectangles: 40 by 10,20 by 30 , and 45 by 5 . He told me that he was really surprised to find that the 45 meter by 5 meter rectangle had the smallest area. So he, Latoya, and I went into a hallway that was about 5 feet wide and looked at how narrow this would actually be.

Travis and Latoya were able to complete the second part of the question pretty quickly. Latoya said that she knew the shape would need to be a square again, since the square from part 1 had a bigger area than the rectangles. They did some calculations and concluded: "It would be better to join the ropes because you can make your width and length wider by each side. By doing this you increase your profit. There is also more land for you and your partner to dig." I was really interested in the comment about profit, and so we talked about it with the whole group. We wondered whether having more land would necessarily guarantee more profit. So in talking about this, we touched on probability, and we also began thinking about what the most equitable way of sharing the plot would be. Is it more fair to split the land, or is it more fair to split the total profit? Most of the students concluded that it would be the most fair to split the total profit, or weight in gold, equally. Though some said they would prefer to take a gamble and have half of the land all to themselves. This was interesting, I thought.


RODOLFO, CLEMENTE, AND JULIO'S APPROACH

Rodolfo and Julio have been with me for a while, but Clemente is pretty new. All have limited English proficiency, and so it was fascinating to me that their description of how they solved this problem was the most verbal of any that I saw. What isn't clear from this photo is that, before they submitted this poster, they had done another that contained pictures and nothing else. I told them that I would like to know more about what they had to say about the problem, meaning that I would like them to talk about it to the group. They decided to start over and take the approach you see above.

After finishing part 1, Rodolfo was sure that there was no way it would be beneficial to work with another prospector. So I asked him to prove it to me, and he started working. When I checked back with their table only three or four minutes later, Rodolfo told me that he was wrong: If he worked with another prospector, he would get twice as much land. Because they answered so quickly, I asked: "What if all four of us decided to join our ropes together? How much land would we get then?" And they produced the explanation on the right. Their drawing is interesting. It suggests that the four small squares could be put together to form the big square with area of 10,000 square meters. I asked them about this. They explained to the group that they didn't mean it that way, and they realized how their drawing didn't accurately represent their thinking. This approach to presenting their solution was great and not at all what I was expecting.

## CRYSTAL AND STEVEN'S APPROACH

Tliked Crystal's approach because of its clarity and simplicity. But it's ـalso worth noting that Crystal needs almost constant support in the classroom. She has a hard time struggling on her own, and her hand shoots up to ask for my help once every five minutes or so. When she first looked at this problem, she gave up right away and said that she didn't have any idea where to start. So we first just talked about what was happening in the problem. Once Crystal figured out that she needed to make a rectangular plot, she was able to produce the four rectangles above. And she worked independently for the next fifteen minutes

without asking a single question. After a few tries, she arrived at the correct answer.

Steven was really struggling. Despite some help and some tips from me, he wasn't able to find any rectangles that had a perimeter of 100 meters. The only one he could come up with was 25 by 25 . This was good, but I wanted to see some flexibility in how he was thinking about this, so I pushed him to keep trying. Crystal was sitting next to him, and when I stepped away to talk to another student, she started showing Steven what she had been working on. Steven followed along with what she was saying and asked her questions. I thought this was a really great moment for Crystal. Here was a student who had no confidence in her own abilities, teaching another student how to create rectangles. I also noticed that Steven was really listening. So I stayed out of the way, and they finished the project together, with Crystal doing most of the heavy lifting and Steven asking good questions along the way.

## Final Thoughts

I really enjoyed doing this problem with my classes, and it's one that I would highly recommend using with any class level. I wasn't whether or not to have small groups present their strategies to the class using posters, but I'm really glad I did. In some cases, I was explicit with students in asking them to represent all of the steps they took to get to their answers-meaning, I wanted to see the mistakes as well as the successes. But with other groups, I just let them go. I found this to be an effective way of structuring the discussion about student responses. By doing this, we got to talk about different ways of structuring and illustrating our thinking, but we also got to talk about the choices that the students made in terms of what to include and what to take out when creating their posters. Over the past two years, a big part of my teaching has involved talking about student work, and this activity only reinforced it for me. Time spent talking about thinking and talking about strategy is just as valuable as time spent solving equations or graphing lines. I also learned a lot about my students' ability to persevere and struggle from doing this activity. I do at least one of these long-form problems every week, and at the beginning of the
cycle, my students tended to give up, get frustrated, and ask me why I was making them do problems like this. But now that we've done ten or twelve of them, my students have become real problem-solvers. It was affirming to see that we can teach persistence and that our students do benefit from it.

## ■ What I Might Change

When I did the problem this time, I asked my students to work independently for about twenty minutes, but then I allowed them to work with the other people at their table for the next forty minutes. I think that this improved the "presentation" element of working on this problem, but I would be interested in seeing what would happen if students just work independently the entire time. I plan to try this next time around, just to see what I get from them. My sense is that the small-group work facilitated some good discussion, and it helped keep struggling students engaged. Even if they weren't able to completely solve the problem on their own, they were able to provide input and feedback as the group worked together. I'm also interested in trying this activity over a period of a week. Students could submit something on the first day. I would then provide some feedback and ask them to clarify their thinking in places, and I would ask them to resubmit their work. I'd like to see how their explanations and processes would change if they were given several days, rather than just an hour, to think and elaborate.

## ■ Unexpected Challenges

I used this problem with two groups of students who didn't have a lot of experience with geometry. Most of them were able to pick up on area and perimeter quickly-in large part because it wasn't completely new-but some had a very hard time. I can think of two or three students who just couldn't figure out how to make a rectangle have a perimeter of 100 meters. Or, if they were able to find one, then they couldn't find one with different dimensions. In these cases, I just asked the students to focus on creating rectangles, not finding the one with the biggest area: "Calculating the area can wait; for now, let's just see how many different rectangles we can find that have the perimeter we're looking for." Next time I'll be better prepared to help students with this part of the problem.

## ■ Student Takeaways

My students really liked this problem, and they liked getting the opportunity to explain how they solved it. The students did learn some important mathematical concepts, but I think that the most
important thing they got out of it was thinking about how they would create their posters so that they could talk about their thinking. They learned that they needed to show the beginning and intermediary steps before just getting to the answer because this would help their classmates understand where the approach came from and how they worked with it. Through the course of the activity, I saw my students start to think like teachers. When talking about their strategies, they explained all their steps and they fielded questions, both from me and from their peers.

## Advice for Teachers

This is a great problem that gives students a lot of material to talk about. On the surface, it just seems like another word problem, but there are lots of extension questions you could pose to encourage further thinking, and there are good discussions that can arise after the students have already found the solution. S o don't feel like you have to rush through it. Take your time, talk to your students about their thinking, and then ask them to show their thinking to their peers. You'll also get a sense of what your students are interested in. Mine, for example, were really interested in turning this into a function (because we had just covered functions). Each student will find something interesting about this problem. So take a little bit of time to let the class go where they want it to go. Your students will appreciate it.

## The Movie Theater Problem

## The Problem:

At a movie theater in Windsor Terrace, the price of a children's ticket is 50 percent of the price of an adult's ticket. Nick and Katie (both adults) took their three children to see a movie yesterday, and the total for all the tickets was $\$ 36.75$. What was the price of each child's ticket?

Please show all your work, and circle your final answer.

## How I Solved It

I solved this problem algebraically. Let $x$ be the price of each child's ticket. An adult's ticket costs twice as much as a child's ticket, so the price of each adult's ticket would be $2 x$. There are two adults, and so the total price for their tickets would be $2 x+2 x$, or $4 x$. There are three children, and so their total ticket price would be $3 x$. The total for all tickets would therefore be $4 x+3 x$, or $7 x$. This should be equal to the amount paid, $\$ 36.75$. So:

$$
\begin{array}{r}
7 x=36.75 \\
x=5.25
\end{array}
$$

A child's ticket costs \$5.25.

## Other Ways to Solve This Problem

One possibility would be guess and check. Let's assume that a child's ticket is $\$ 6.00$. If this were true, then the total for the children would be 3 times $\$ 6.00$, or $\$ 18.00$. Since the adult tickets cost twice as much, and there are two adults, the adults paid $\$ 12.00$ times 2 , or $\$ 24.00$. The total for this scenario would be $\$ 42.00$, which is too high. I could keep adjusting my guess until I get the correct result.

I could also put this information into a table, which would help me to organize my guesses. Let's say I started by guessing that a child's ticket is $\$ 4.00$, and then I recorded each subsequent guess into the table. It might look something like this:

| Child's Ticket | Adult's Ticket | Total Cost |
| :---: | :---: | :---: |
| $\$ 4.00 \times 3=\$ 12.00$ | $\$ 8.00 \times 2=\$ 16.00$ | $\$ 12.00+\$ 16.00=\$ 28.00$ |
| $\$ 5.00 \times 3=\$ 15.00$ | $\$ 10.00 \times 2=\$ 20.00$ | $\$ 15.00+\$ 20.00=\$ 35.00$ |
| $\$ 6.00 \times 3=\$ 18.00$ | $\$ 12.00 \times 2=\$ 24.00$ | $\$ 18.00+\$ 24.00=\$ 42.00$ |
| $\$ 5.50 \times 3=\$ 16.50$ | $\$ 11.00 \times 2=\$ 22.00$ | $\$ 16.50+\$ 22.00=\$ 38.50$ |
| $\$ 5.25 \times 3=\$ 15.75$ | $\$ 10.50 \times 2=\$ 21.00$ | $\$ 15.75+\$ 21.00=\$ 36.75$ |

## Why Did You Choose This Problem? What Do You Like About It?

I like the problem because even though the situation is easy to understand, solving the problem is difficult. Because my students aren't familiar with algebra yet, they'll have to puzzle it out using a bit of guessing and exercising precision and control over basic operations. I've noticed throughout the cycle that this group of students does very well with calculation and that they don't have too much trouble working out rather complex problems when the methods required are clear to them. However, they struggle when they have to puzzle things out for themselves, especially when there are no given answers to choose from. Had I given this problem the first week of class, I probably could have expected most of my students to come up with an incorrect answer. Now that we've had a bit more practice with problem solving, though, I expect to see far more correct answers. And finally, this question requires close reading, which is what my students struggle with the most. I deliberately made the wording in the first part a little more confusing than it needed to be, which will require them to think carefully about how the two ticket prices are related. They'll also be thrown by the percent figure in there, and I'm guessing that a number of students will try to apply formulas like the percent proportion, which probably won't be of much help here. The question requires a more comprehensive understanding of what percent means. I'll also be interested to see how many "correct" answers I get that give me the price of an adult's ticket instead of a child's ticket.

## Why Is This a Good Problem?

I think this problem is a good fit for this particular group, and it will help my students to persevere in problem solving. The problem has at least a couple of different solution methods, and it requires a synthesis of basic operations, close reading, and checking the answer. It requires several steps in that the students will have to correctly identify the number of adults and the number of children and then perform a few
rounds of basic operations. It requires also reasoning and inference in that the students should use their own knowledge of the world to assume that a child's ticket likely wouldn't cost something like $\$ 1.50$ or $\$ 22.00$; common sense should help the students to arrive at an answer. Students will definitely have to think about what the question is asking before jumping right in, because in a sense they'll be working backward from the total to determine the cost of the "parts." I think that this requires a holistic approach to the concepts, in that they'll see how the basic operations relate to each other and to common, everyday settings. This question also requires careful reading, and most students will need to read it at least two or three times to really get a handle on it . Lastly, this problem definitely invites the use of a few different problem-solving strategies. I'm assuming most students will use guess-and-check; they'll draw pictures; they'll work backwards; and they'll look for patterns as they try these strategies.

## Challenges for Students

## - Reading

As I noted above, my students have more difficulty with understanding the question than with performing calculations. I anticipate that some will simply assume all the ticket prices are the same; others might misinterpret the number of people seeing the movie; and others might return the price of an adult ticket instead of a child's ticket.

## - Puzzling out an answer

This group has a hard time digging into really difficult problems when the answer choices aren't given to them. I think that this problem will make them try a few different methods or guesses to arrive at the answer. Some might get frustrated in doing this and might just give up.

## - Checking their answer and making sure it makes sense

As with most of the students I teach at this level, this group has a tendency to come up with an answer and then just go with it, without making sure that it makes sense or that it actually checks out. When I see the student work, I'm hoping to see that at least a few people went back and made sure their answer was correct.

## Support

- I will emphasize careful reading and ask good clarifying questions. If a student is struggling, I'll read through the question with them and make sure that they understand both the criteria set forth in the problem and exactly what it is that the question is asking them. This will help to set them on the right track in terms of figuring out how to solve the problem.

■ "I don't know where to start." This is something that I hear a lot, and when this happens during this activity, I'll ask what the student thinks the answer could be. So, if a student is really struggling with this problem, I'll ask that they start by taking a guess at what the cost of a child's ticket will be, then I'll see where that takes them.

- I will need to help students think about their answers and whether or not they make sense. I plan to ask students how they can be sure that their answer is correct and asking that they prove their answers to me. This will show me that they really understand the question and why their answer is correct.


## Student Work

## LINDA'S APPROACH

Linda is perhaps the brightest student in the class, but because her English-speaking and reading skills aren't too good, she sometimes gets hung up on problems with confusing wording. Her work on this exercise exemplified this problem. I knew when I included a mention of "50 percent" in the question that students would first try to use the percent proportion to solve; this was Linda's approach. She found that 50 percent of 36.75 was equal to 18.375 , and it was hard for her to let go of this number. Once she got it, she divided it by three and assumed that each child's ticket must be $\$ 6.125$. She was then convinced that this had to be part of the answer. Linda's work-and her failure to solve the problem correctly-showed both me and her how much she needs to work on creative approaches to problem solving. Linda immediately associated the percent figure with the percent proportion that we worked on in class, and then couldn't understand how it could be wrong. She feels comfortable working with formulas and struggles when they don't always work out.

## 1 Why did you choose this sample of work?

As I mentioned above, this sample shows a problem that Linda has struggled with all semester. She doesn't like to take guesses, draw pictures, or work backwards. She likes to fall back on the formulas that she has memorized, and when those don't work out, she freezes and can't understand how her answer could be wrong, no matter how much I try to lead her into thinking about the problem more comprehensively. Linda's work shows a student who sees numbers, plugs them into what she knows, and thinks that it should result in the correct answer, which, most of the time for Linda, it does. Oddly, Linda also started asking people across the room for clues, which is something I've never seen her do. When I would turn my back to her and work with someone else, I would hear her whispering to other students asking how they got the answer. It really shows how uncomfortable she gets when her formulas don't work.

## 2 How typical was this student's approach in your class?

Very typical. Of the seven students who tried the exercise, all seven started by finding 50 percent of $\$ 36.75$. Most of the others, however, realized that this approach wasn't getting them anywhere and then moved on. Linda had a hard time with this, even when I changed the wording of the question to say that "an adult's ticket cost twice as much as a child's ticket." Although I thought that changing the wording might affect Linda's thinking-which it did for many of the other students-she still couldn't shake the notion that 50 percent of $\$ 36.75$ had something to do with the answer.

## RUBEN'S APPROACH

From Ruben's work, it's difficult to tell exactly how his thinking is organized, which I found interesting. He began by dividing 36.75 by 5 , and came up with an answer of $\$ 7.35$. When I pressed him on that answer and asked if everything checked out, he was confident that it did. But then when I asked him if the adults and the children all paid the same price for tickets, he recognized his mistake. Then, like Linda, he found 50 percent of the total ticket cost, but he seems to have realized fairly quickly that it wasn't working. Unlike Linda, though, he continued to try to puzzle out an answer by labeling different numbers and by repeating values for the ticket prices for the adults and for the children. Ruben was not able to come up with the correct answer.

## 1 Why did you choose this sample of student work?

I primarily chose it because of the student. Ruben isn't very strong mathematically, but in general he has a good head on his shoulders and enough real-world experience using math that he's able to come up with correct answers, or at least strong guesses. In two consecutive classes, he passed both both forms of the TASC Readiness Test. This surprised me, in large part because Ruben's in-class work usually looks like it did on this exercise; that is, he tends to just jump into a problem and plug in numbers. So I found this example of work interesting because it seems like Ruben doesn't really have a plan, but he has consistently scored well on all assessments he's given in class, which tells me that I must be missing something. This kind of disorganized work is something that I warn students against in class, but Ruben doesn't seem able to break the habit-and maybe that's okay. To my mind, organization seems essential to doing math, and through his work Ruben is making me call that idea into question. Or maybe we just have different ideas of organization.

## 2 How typical was this student's approach in your class?

Fairly atypical. In most of the samples I collected, the students seemed to really consider things for a long time before writing anything down, and even then, they didn't write much. Ruben just went right for it and worked with just about any number he could find. By far, he wrote more and tried more than any other student.

## ARI'S APPROACH

Like Ruben and Linda, one of the first things Ari did was calculate 50 percent of the total ticket price. She appears to have abandoned that idea pretty quickly though. She then did what Ruben did and divided 36.75 by 5 to get $\$ 7.35$. This is where Ari made an interesting mistake: she began calculating 50 percent of 7.35 , and then tried to work with the resulting value, $\$ 3.67$, as the child's ticket price. At this point she got stuck and asked for some guidance. When we checked her answer against the criteria set forth in the question, she saw that her answer was wrong and decided to try something else and started guessing. As you see at the bottom of her page, she tried a few different child's ticket prices and then calculated the totals, adjusting her guesses as necessary until she arrived at the correct answer.

## (1) Why did you choose this sample of student work?

Ari was the only student who answered this question correctly on her own, and she was the second student to come up with an
answer after Ruben's initial incorrect one. In all, it took her about ten minutes. Everyone else in the class labored over the problem for more than a half hour and needed a good deal of guidance to come up with an answer. When Ari got the answer of $\$ 5.25$, she was also confident in it and was able to prove that it was correct using her calculator. Ari came into the class pretty unsure of herself, but she has become a great student and a leader in the classroom. Her ability to try different methods and be confident in her answer shows really reflects her growth as a learner.

## (2) How typical was this student's approach in your class?

Initially, it was very typical. Ari was hung up on the numbers given to her in the problem and fell back on the mathematical methods that she would normally use to solve a problem. But when she realized that those didn't work, she felt comfortable stepping away from them and making a guess. Soon after her first incorrect guess, she realized that she would get to the correct answer using the guess-and-check strategy and continued working until she came up with the answer. Very few other students felt comfortable guessing, and even when I recommended that they try it, those students were resistant.

## Final Thoughts

I learned that even though I have adopted more of a problem-solving approach to the teaching of math in this course, I still have a ways to go. My students' relative inability to puzzle out an answer to this problem showed me that they still have a long way to go in terms of being problem solvers. They did a good job mathematically, which shows me that I've at least done a reasonably effective job of teaching computation, but their work on this problem evidence a real lack of comprehensive understanding of mathematical concepts-in this case, percent. I mentioned this in my workshop reflection, and it was made concrete in class when I gave my students this problem.

All of this showed reminded me that teaching computation is such a small part of the battle in HSE math. I'm now thinking that the best thing to do is to work on fewer problems in class and instead really work hard on just a few problems, emphasizing the deep connections between concepts. It's great for students to see and work on a wide variety of problems, but if the conceptual understanding is missing, they still won't be able to solve difficult problems like this one on their own.

■ What I Might Change
I think that in changing the wording to involve a mention of percent, I made this problem a little too difficult. Next time, I might try saying that an adult ticket costs twice as much as a child's ticket and then see where my students take it. Another thought I had is that we could have a discussion as a class about what the 50 percent means in the context of the question. Does it mean that the children's tickets were 50 percent of the total cost? How can we connect 50 percent to a fraction? What do we need to calculate 50 percent of? And so on.

## ■ Student Takeaways

I think that there is a great benefit in applying something that you know you know only to learn that it doesn't always work out the way you think it should. In this case, almost everyone jumped straight to the percent proportion, because it so often does help them to get to the correct answer, even if it's only a step along the way. Here, falling back on the formula they had been using all semester actually did them a disservice in solving this problem. It is my hope that working on this problem for the length of time that they did helped them to understand that they sometimes need to be a little bit more creative.

It also really emphasized close reading, which is something that all of my students have identified as something they need to work on. Several students came up with answer that added up to $\$ 36.75$, but they ignored the criteria that a child's ticket costs half as much as an adult's ticket. While it's important to know what the question is asking and to focus your efforts toward answering it, it's also important to keep the constraints in mind.

## Unexpected Challenges

I really didn't think they would have quite as much trouble working with and understanding the constraint built into the problemthat children's tickets cost half as much as adult tickets. Several of them were able to come up with answers, but no student-save perhaps Ari-showed me a correct answer on their first try. It was also difficult to nudge students away from calculating 50 percent of the total ticket price without giving too much away. And then when I did, and when I encouraged them to try making a guess, they would kind of roll their eyes and dismiss the suggestion, because I think that many of my students see guessing and checking as "not real math." They want to know a more concrete, more typically "mathematical" method. So this is something that I will try to build on more before I give this question in the future.

## Advice for Teachers

I would recommend that a teacher first give the problem exactly as it is written, just to see what students are able to com up with. But then, I think it's important to carefully structure the clues given to the students as they continue struggling, and it's just as important to reinforce problem-solving strategies. I might also recommend leading with a less confusingly worded constraint; the teacher could then discuss other ways that the constraint could be written. For example, each student could say that a child's ticket is half as much as an adult's, or that each adult's ticket is twice as much as a child's, or that an adult's ticket costs $100 \%$ more than a child's ticket. This might be an interesting way to reinforce concepts.

Guessing and checking got a few people to the right answer, but that was the only method that worked. After the first twenty minutes to a half hour, I put five answer choices on the board and told them that one of those answers was exactly correct. At this point, the students started to remember that they could try each one against the constraints set forth in the question-which, at this point, we had gone over together as a group to make sure that everyone was on the same page-and they came up with the answer. I was pleased to see that they were able to work backwards, but when I teach that solution method next semester, I'll be sure to do a better job of linking it to guessing and checking.

It might also be interesting, once students have arrived at the correct answer, to have them create a similar question of their own and then solve it. Or it might also be good to have students write new questions based on the values in the question and the answer. I like to emphasize that doing math is a creative process, and I often have students write their own problems based on the concepts covered in class that day. I haven't, however, had them use information from an existing question-one that they had solvedto come up with something unique. I think I'll try that next time.

# 66 Education is what happens to the other person, not what comes out of the mouth of the educator. 99 <br> -Myles Horton 

