3

PATTERNS AND FUNCTIONS

From A Collection of Math Lessons, by Burns & Humphreys

California's *Mathematics Model Curriculum Guide, Kindergarten Through Grade Eight* (1987, 35) states the following:

> Looking for patterns helps bring order, cohesion, and predictability to seemingly unorganized situations. The recognition of patterns and functional relationships is a powerful problem-solving tool that enables one to simplify otherwise unmanageable tasks and to make generalizations beyond the information directly available.

This chapter presents activities that make ideas about patterns and functions accessible to students. Eight days of instruction with a class of seventh graders are described.

Students were first introduced to the idea of growth patterns and then used Color Tiles to create patterns of their own. Students analyzed other students' patterns and made predictions about how these patterns could or "should" grow. After an investigation of how increasingly larger squares grow, the students engaged in several investigations with cubes. In all activities, students were able to demonstrate and verify their thinking with the tiles and cubes. The materials were tools for making ideas about patterns more concrete.

Though students were introduced to writing formulas to express the generalizations of functional relationships, I kept the emphasis on their concrete explorations of building growth patterns and analyzing them geometrically and arithmetically.

Day 1

To prepare, I assembled eight plastic bags of Color Tiles with about 100 tiles in each. I had these on the counter ready to distribute to each group of four students.

"What comes to mind when I ask you to think about pattern?" I asked to begin the class. "Talk about your ideas in your groups and have one person record what you discuss." It was quiet for a moment before their discussions started.

After a few minutes, I called the groups back to attention. "Choose someone from your group to report one thing from your discussion," I said. I gave them a minute to do this.

Mark was first. "Our class schedule is a pattern," he said. "Like, it goes 1, 2, 3, 4, 5, 6, and the next day it goes 6, 1, 2, 3, 4, 5. We end today with fourth period, and we start tomorrow with fourth period." Our school is on a rotating schedule, so classes meet in a different time slot daily.

Teddy reported next. "The week is a pattern," he said, "Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday."

Pam reported for her group. "There are patterns in clothes," she explained, "like when you have stripes, like on my shirt."

"How many of you have patterns in the clothing you're wearing today?" I asked. Students looked around at what everybody was wearing. Quite a few raised their hands.

Sheila then reported for her group. "Wallpaper has patterns," she said.

"Piano keys have a pattern," Aurora said, "like white, then black, white, then black."

"Daily life has a pattern," Richard said. "You go home, cool off, eat, play, do homework, eat dinner, watch TV, brush your teeth, and go to bed. It's a rotating order."

"All the patterns you reported," I said, "are examples of things that repeat over and over. The pattern makes it possible to predict what will come next."

"Is that what patterns are?" Tami asked, "things that repeat?"

"Things that repeat are one kind of pattern," I answered. "Once you understand a repeating pattern, you can tell about what comes next. For example, even though wallpaper stops at the end of the wall, you can describe what would come next in the design. It's predictable."

I was interested, however, in focusing the students on patterns that grow rather than patterns that repeat. Since none of their examples described growth patterns, I introduced this idea with two examples.

"Mathematicians rely on patterns to make predictions in many sorts of problems. I've planned a collection of activities to help you learn how mathematicians use patterns to make predictions. In these activities, however, the patterns aren't ones that repeat. Instead, they're patterns that grow."

"Like how?" Nguyen asked.

"Here's an example," I said. "There's an old favorite math story about a boy who agreed to wash the dishes every night to earn some money. He said he'd charge just a penny."

"No way," James said.

"That was just the first night," I continued. "He said he'd charge \$.02 the second night, then \$.04 the third night, then \$.08, and so on. Each night he doubled his fee."

"Oh, yeah," Natalie called out. "I heard that story. He makes a lot of money, like millions or more in a month."

"That's impossible," Paul said.

"Let's figure it out," Mark said.

"How would you figure it out?" I asked.

"It's easy," Mark answered. "You just keep doubling."

"That's an example of a pattern that grows instead of repeats. He doesn't earn the same thing over and over, but you can predict what he'll earn the next night because of the pattern of doubling."

"Can we figure it out?" Aurora asked.

I hadn't planned to take class time to have them do this, but because their interest was high, it seemed best to let them satisfy part of their curiosity.

"Figure out how much he would be paid on the tenth night of washing dishes," I said. "Check with others in your group to see if you agree."

After a few moments, the students had come to consensus that he would be paid $5.12 on the tenth night.

"Not bad for washing dishes," Greg said.

"But it's not millions," Paul said.

"I noticed that most of you listed each day's salary to figure out how much he earned on the tenth day," I said. "You know the pattern, so you can keep going. But I don't want you to do more figuring now. Right now I want to focus on helping you understand more about how mathematicians use growth patterns."

I waited until I had everyone's attention.

"Here's another example of a pattern that grows," I said.

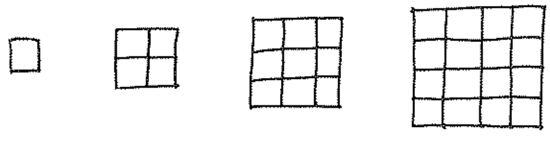
I showed them an article titled "Faces from the Future" that appeared in the February 13, 1989 issue of *Newsweek* magazine (Cowley and Springer 1989, 62). There were three pictures with the article: a photograph of an eight-year-old girl, a computer-generated portrait of the same girl as she might look as a fourteen-year-old, and an actual photograph of her taken at age fourteen. The computer picture was generated from the photograph of the girl as an eight-year-old. It had been used successfully to find the girl six years after she had been abducted by her father. The similarity between the computer portrait and the actual photograph is impressive.

I reported to the class from the article. "The article explains that changes in 39 facial measurements were used by the computer," I said. "The measurements included the length of the nose, the width of the mouth, and the space between the eyes. The article says, 'The seeming magic all stems from the fact that facial bones change in predictable ways throughout childhood.' For example, the article explains, 'the distance from the bridge of the nose to the base of the chin typically increases 12 percent between the ages of 6 and 13.' " The students were intrigued by this information. Some made comments. "That's neat." "Can I see those pictures?" "How do they do that?" "They're really close."

"In this situation," I told the class, "being able to make predictions about patterns of growth helped solve a kidnapping case. Making predictions from growth patterns is an important area of mathematics that we'll be studying for a while. For the rest of class today, you're going to experiment with growth patterns using Color Tiles."

In the past, I've begun instruction about patterns by giving students patterns to investigate. However, I've come to learn that when students understand the purpose of what they're being asked to learn, they are more motivated. Also, an introduction such as this can add to students' understanding of the role of mathematics in our society.

I then drew a sequence of four squares of increasing size on the overhead.



"Why do you think I would call this sequence of squares a growth pattern?" I asked the class.

"Because they're getting bigger," Tina said.

"They're like just enlargements," Russell said.

"Suppose I want to draw the next square in the sequence," I said. "What would I draw? Talk in your groups about this. Then I'll have someone give me directions to draw the square that comes next."

I gave them a few moments to talk together. After hands were raised by all but two groups, I called on Kimberly.

"It would be a square that has five small squares across the top and bottom and down the sides," she said.

I drew a five-by-five square on the overhead. "Yes, this is the fifth square in this pattern," I said. "I could continue the pattern by drawing increasingly larger squares. This is an example of a growth pattern that's fairly obvious."

I then drew a small square on the overhead. "I'm going to start another pattern the same way I started the square pattern," I said. The next idea is an adaptation from an activity in *Unit I: Seeing Mathematical Relationships* (Bennett, Maier, and Nelson 1988), the first of six units in the Math and the Mind's Eye materials.

"This is the first in the pattern. When I add squares to make it grow to the second in the pattern, I get this." I drew an L-shaped pattern using three squares.

"It may not be as obvious as it was with the squares what the next pile would look like," I said. "As a matter of fact, there are several different ways to continue the pattern of growth, so my next question doesn't have just one right answer. What might the next pile look like?" Several students raised their hands immediately, but I didn't call on any of them.

"I don't want to hear your ideas yet," I said. "Instead, you each are to use one color of tiles and build the two figures I've drawn on the overhead. In a moment, one person will get a plastic bag of tiles for your group. When you've built what I've drawn, think about how these might grow into the next pile and use more tiles of the same color to build what you think the third pile might look like. Then, using the same pattern of growth, build the fourth pile and cover it up with a sheet of paper. While you're working, be thinking about how you can explain how your piles grew."

I gave one more direction. "Before you get started building your individual patterns," I said, "talk in your groups to make sure everyone understands what to do." I had given a hefty amount of information, and I know that some students have difficulty following verbal directions. I felt that having them talk among themselves would help orient all of them to the task.

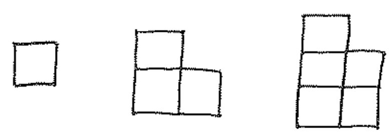
After a few minutes, I called the class to attention. "Share your growth pattern with the other members of your group," I said. "See if you can predict each other's fourth piles."

Some groups had two or three different patterns. In some, everyone had the same pattern—the L. There were some students in the class who had no idea how to make third or fourth piles.

Students use tiles to show how the pattern might grow.

After the groups had had time to look at the patterns group members had made, I interrupted them. "Who can describe a third pile for me to draw?" I asked.

Denise volunteered. "I built a pile with a row of three going up and down, and right next to it a row of two going up and down." Denise's group confirmed that I drew it correctly.



"How did Denise add tiles to the second pile to make it grow?" I asked.

"She added one on the top left and one on the top right," Kimberly answered.

Greg had a different idea. "I thought she added two on the bottom," he said.

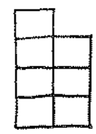
"Either way seems to produce the same result," I said. "There often are different ways of looking at the same thing. How might you describe what this third pile looks like?"

"It looks like a building with a chimney," Norberto said.

"It's like a building with a room knocked off," Teddy said.

"Now build what you think the fourth pile in Denise's pattern would be," I then said.

The students did this quickly. I drew on the overhead what I saw on the desks and Denise's group confirmed it was correct.



"Suppose we continued this pattern," I said. "Can you predict what the twentieth one in the pattern would look like? Close your eyes for a moment and try to imagine its shape. Raise your hand when you think you can see it."

I purposely didn't ask how many tiles would be used for the twentieth pile or how tall it would be. I wanted them to focus on the idea of growth geometrically. I've found that focusing on number often moves the investigation from looking at growth patterns to getting the right numerical answer. There would be time later for numerical interpretations.

I waited until more than half the class had raised their hands. I then called on several to describe what they had envisioned.

"It's more like a tall skyscraper with two columns," Tami said.

"It's a high building with a short chimney," Paul said.

"It's still a building with a room knocked off," Teddy said.

"Did any groups build a third pile that's different from the one Denise described?" I then asked. Several students raised their hands. It was just about the end of the period.

"I'm interested in your other patterns. We'll start tomorrow with this again," I said. "It's time to put away the tiles. Also, it may be a good idea for someone in your group to make a sketch of your third and fourth piles if they're different from what Denise described." There was the usual bustle of getting ready to leave.

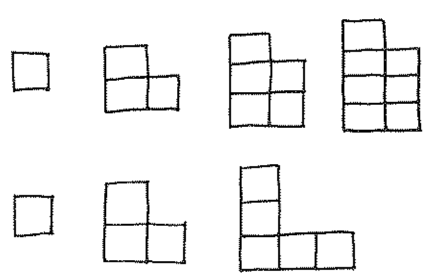
Day 2

I drew the first four piles in Denise's pattern on the overhead as a reminder of what we had been doing yesterday. Also, I distributed the Color Tiles.

Underneath Denise's pattern, I drew the first two piles again so that other third and fourth piles could be added and compared with what Denise had built.

"Who can come up and draw a third pile that's different from what your group discussed yesterday?" I asked.

Richard came up and drew an L shape.

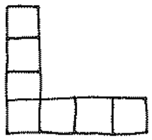


"That's like ours," Conroe said.

"Ours, too," Kimberly said.

"Quickly take out some tiles and build what the fourth pile will look like," I said. Everyone in the class was able to build the next in the pattern.

After the students had time to build the fourth pile, I drew it on the overhead.



"How does this pattern grow?" I asked.

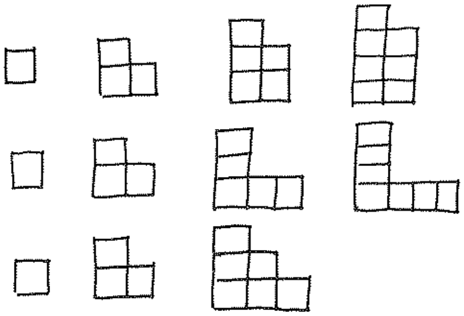
"You just add one to each end," Pam said.

"You get a bigger L each time," Conroe said.

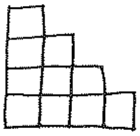
"There are two lines with an overlapping corner," Patty said.

"Show me with your hands," I said, "what the twentieth pile will look like." Some held their arms out to make an L. Others used just their hands. Some sketched an L in the air.

"Does any group have another growth pattern?" I asked. I drew the first two piles again. Teddy raised his hand and came up to draw another third pile.



"Now build what the fourth one will look like," I said to the class. They did this fairly quickly. Their confidence seemed to be increasing.



"Who can describe how you built the fourth pile?" I asked.

Ron raised his hand. "I saw a slanted row of two added to the first pile, and a slanted row of three added to the second pile, so I added a slanted row of four to the next pile." I drew the fourth pile, adding slanted rows as Ron described.

Richard described a different way. "I built a vertical row of four, then of three, then of two, then one," he said.

Pam saw it differently. "I just added another row on the bottom that had one more in it," she said.

"What would the twentieth look like?" I asked. Hands shot up.

"A big staircase," Aurora said.

"A long, jagged edge, like teeth," Conroe said.

I then introduced an activity for them to do in pairs. "Now you'll have the chance to build growth patterns that you invent. You'll work in pairs, with the person seated next to you. Together, build four piles in a growth pattern using Color Tiles."

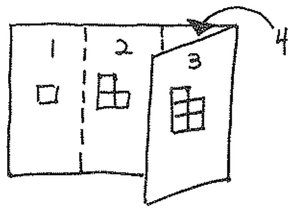
"Do we start with one tile like you did?" Tami asked.

"You can," I answered, "but you don't have to."

Some students began to get to work. I interrupted them to give further directions.

"Record your work so others can look at the first three in your pattern, then predict the fourth," I said.

Tami raised her hand. She had an idea about how to record. "If we fold the paper into four columns," she said, "then you can fold one flap over and hide the fourth drawing underneath." Tami showed the class what she meant.



Her suggestion made good sense. "Let's do it that way," I said. "When you're done, you'll have the chance to trade papers and see if you can figure out others' growth patterns."

The class got right to work. I helped those who had difficulty getting started. The students were all involved and seemed interested. A few partners finished in time to trade papers. Most completed the assignment, but a few needed a bit more time. I collected their papers at the end of the period.

That night, I examined their work and sorted their papers into three groups. There were seven that were complete and had predictable growth patterns. There was one that was complete but had an error in the fourth drawing. Five weren't finished. I decided to structure the next day's lesson so that I could work with students who needed help while the rest examined the patterns made by other groups.

Day 3

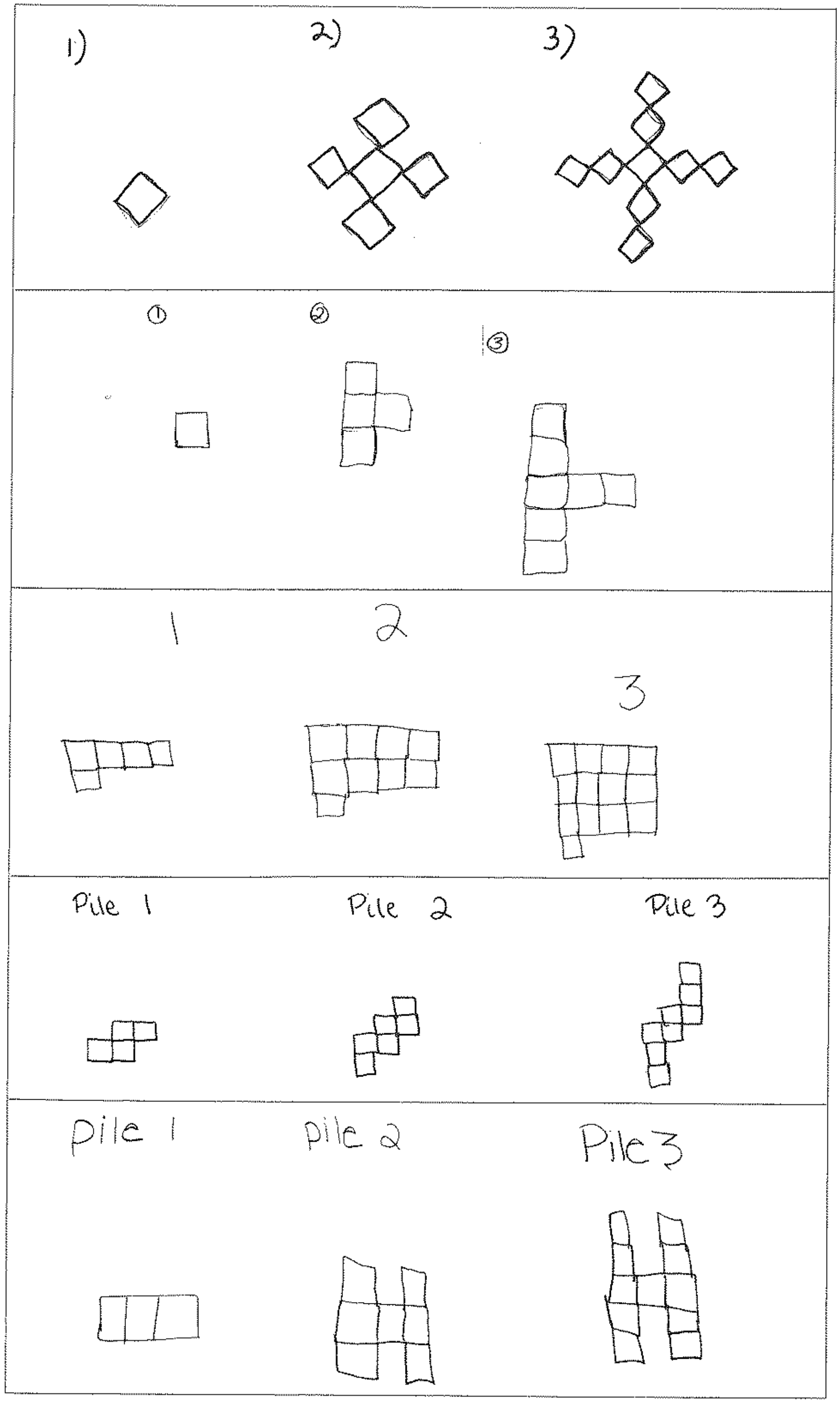
I began the period by explaining how the students were to analyze others' patterns. "Use a half sheet of paper for each pattern you analyze," I began, "and write your names and the names of the people who created the pattern. Then look at their first three drawings and together predict what the fourth drawing would look like. Use the tiles to build it or make a sketch. Then check to see what they drew.

"After you check your prediction, describe how the pattern grows. Finally, draw what you think the fifth pile will look like."

"What about if we didn't finish our own patterns yet?" Norberto asked.

"I've separated the unfinished papers, and I'll pass those back first," I said. "Then I'll give the rest of you someone else's pattern to analyze."

I then handled some of the management details. "We'll keep the papers organized in three piles up here at the front of the room. The first pile is for finished patterns ready to be analyzed by others. The second pile has half sheets of paper for you to use. The third pile is for your completed analyses."

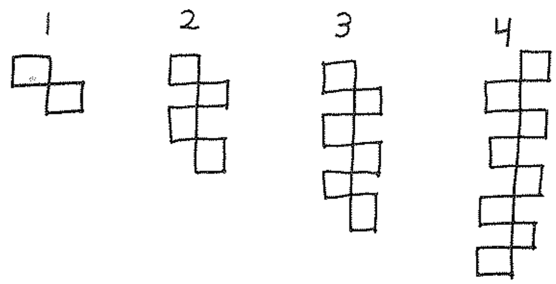


Students created a variety of growth patterns.

I circulated through the room distributing the unfinished papers. "When you finish," I said, "put your paper on pile 1 and take someone else's pattern to analyze."

Then I randomly distributed patterns for the others to analyze. "I've prepared three extra growth patterns for the first pile," I said. "That way, we won't run out if some of you work more quickly than others." I distributed forms to these students as well.

Finally, I returned Jessica and Roger's pattern to them to talk about why their fourth pattern didn't make sense to me. I showed them their first three drawings and asked them to explain their growth pattern.



"We just added 2 more each time like this," Jessica said, pointing to 2 squares with corners touching.

"How can you explain your fourth drawing?" I asked. They immediately saw the error they had made. Instead of adding 2 more to an end of the third pile, they had split them and added a tile to each end. They saw how to correct their paper and got to work.

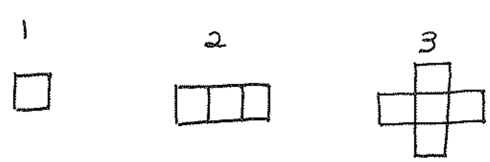
Mark and Hal were arguing about their unfinished work. There were three drawings on their paper, and they needed to add the fourth. I asked them to describe the growth pattern they were trying to show.

"It doesn't make sense to me," Mark said.

"He didn't like my idea," Hal said.

"What was your idea?" I asked.

"Well, I thought we'd add some on the side and then some on the top and bottom," he said, showing me the first three drawings.

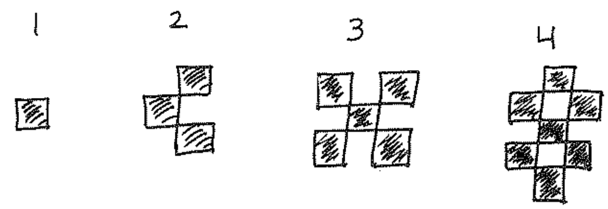


"But you have to do the same thing," Mark said. "You can't just keep changing."

"But I wanted to make it harder," Hal said.

"I have a suggestion," I said. "Hal, continue with your idea and draw the fourth pile. Then Mark, see if you can figure out what Hal is thinking and make a fifth pile with the tiles. What Hal is doing is complicated, but I think it is predictable and reasonable."

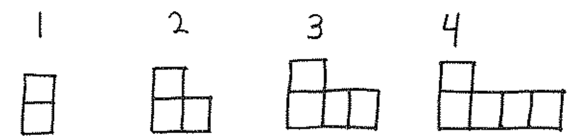
I then talked with Brett and Richard. They had a pattern for which the number of tiles they used grew predictably. They explained to me that they had used 1, 3, 5, and then 7 tiles.



"I see that," I said, "but I don't see any orderly way the arrangements of the tiles are growing. I can't predict what the next will look like." The boys decided to abandon their plan and start again.

As with most assignments, there was a range of responses. For some students, the assignment was easy, and they whipped through analyzing several patterns. When I checked what was being put in the pile of completed work, however, I returned several for more explicit explanations. For others, the assignment wasn't as simple, but it was appropriate. There were a few students for whom the idea of growth pattern still didn't make sense.

I gave the class the homework assignment of analyzing a pattern I had created. I drew the first four in my pattern and had them copy my sketches. "For homework," I said, "analyze my pattern as you've been analyzing each other's. Tomorrow you'll begin class by comparing your work."



That night, I reviewed their analyses of their classmates' patterns. Students took different approaches when describing them. Some focused on the patterns visually and described their shapes.

They're like stairs with more and more steps each time.

It would always have a line of squares with one sticking out.

They have four branches and every other one the branches are the same and then the other one two branches are different.

Other students did not describe the shape of the patterns but gave numerical analyses.

You just double the number for each one.

They added two tiles in each pattern so you go plus two each time.

Some students' descriptions included both geometric and arithmetic analyses.

The two rows keep going up vertically. They added one to each row each time.

It's always an odd number in a row plus one sticking out.

After looking at and thinking about their work, I decided to focus on encouraging all students to look at patterns both geometrically and arithmetically.

Day 4

While the groups compared their homework at the beginning of class, I checked that individuals had done the assignment. The homework had been easy for the students.

I then gave them an additional challenge. "Discuss in your groups what the twentieth pile would look like and how many tiles I'd need to build it," I said.

I could tell that this also was easy for the class. There was some quick discussion and then around a dozen students raised their hands. I called on Paul.

"There would be a row of 20 tiles with one on the top on the left, so it would have 21 tiles," he said. There were nods of agreement.

"What about the hundredth pile?" I asked. Lots of hands went up. I called on Brett.

"There would be 101," he said, "a row of 100 and one extra."

"What about the thousandth pile?" I asked. Again, there were lots of hands. Natalie explained that there would be 1001 tiles.

"What about if I wanted to know how many tiles I needed for the forty-third or hundred thirty-seventh, or any other pile?" I asked. This also was obvious for them.

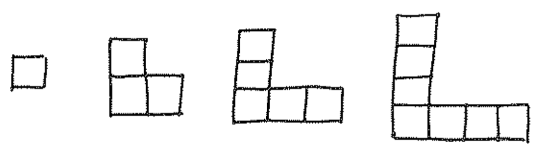
"You just add one," Richard said, "because you'll always have a long row with one on top."

"Here's how mathematicians might describe my pattern," I said. I wrote "$n + 1 = t$" on the overhead.

"You take the number of the pile," I said, pointing to the n in the equation, "and add one on top and that gives you the number of tiles." I pointed to the t.

I've learned from past experience that representing growth patterns algebraically moves the investigation into an abstract realm that is often out of the students' experience. Though I wanted to introduce the idea that a formula can describe a growth pattern, I gave this explanation with a light touch. It seemed to make sense to them, but I didn't emphasize it or push it any further. My goal wasn't to get them to write formulas; rather, it was to focus them on investigating patterns numerically and spatially.

"Let's try another," I said. "Here's a growth pattern we talked about the day before yesterday." I drew it on the overhead.



"Oh yeah, the L," Conroe said.

"Talk in your groups about how many tiles would be needed to build the twentieth L," I said.

This wasn't quite as obvious as the homework pattern, but after a few minutes about half a dozen hands were raised. I gave the groups a few more minutes to work and then brought the class to attention. I called on Kimberly.

"We figured 39," she said. There was a chorus of agreement.

"Explain how you got it," I said.

"Well, it's 20 up and then 20 across," she said, "but you can't count the corner twice, so you subtract 1."

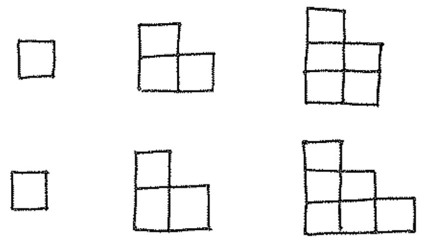
"You did the number of the pile plus the number of the pile and then subtracted 1," I said. I wrote "$n + n - 1 = t$" on the overhead. Again, I treated the formula incidentally, as a way to record what Kimberly had reported.

Richard's hand shot up. "We did it another way," he said. "We didn't count the corner, so there were 19 on the two pieces. Then we added the corner last."

Aurora had another way. "We just added 20 for the vertical," she said, "and then 19 for the flat side."

"Hearing your different methods is a good reminder that there's more than one way to look at a problem," I said. "You should always look for another approach even when you think you've found the answer."

I then drew on the overhead the other two patterns we had investigated the day before yesterday.



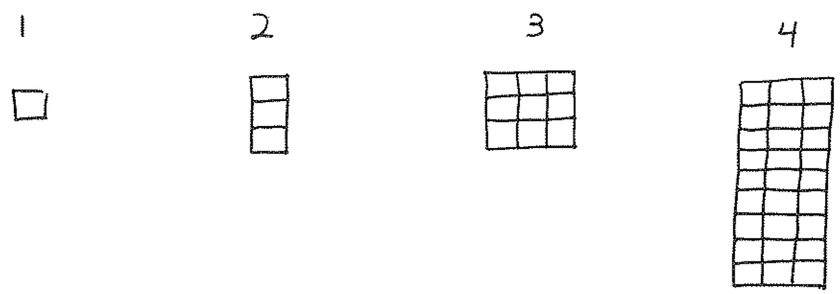
I had the students work in their groups to consider the twentieth pile of each. "Decide how the twentieth pile will look and how many tiles are needed," I said.

I circulated as the groups worked. They were focused and their discussions were animated.

Pedro came up to me. "Look," he reported, "the towers are just like the L's except the short side is up instead of flat." He was pleased with his discovery.

I interrupted the class when all groups had figured how many tiles were needed for the first pattern and most had completed the second as well. About ten minutes remained in the period.

I sketched the four drawings from Pam's and Greg's pattern on the overhead. A few of the students in the class recognized it as one they had analyzed the day before.



"Discuss this growth pattern in your group," I said. "Don't figure how many are in the twentieth pile. Just discuss how it's growing." There was an immediate buzz in the class that died down in a moment. Five hands were raised. I called on Huy.

"It just goes times by 3," he said.

"That's what we got," Aurora added. "1, 3, 9, 27."

I called on Natalie next. "The next one would be a square, because it goes square, column, square, column," she said. Others nodded.

"What would be the dimensions of the square in the next pile?" I asked.

There was some discussion among groups. Several hands were raised. I called on Rodney.

"It would be 9 by 9," he said.

"How do you know?" I asked.

"I don't know," he said at first. I waited a moment. Rodney then added, "Well, the first square was 1 by 1, and the second one, you multiplied by 3 to get 3 by 3, so I just multiplied by 3 again."

"What do you think about Rodney's prediction?" I asked Pam and Greg. They smiled and nodded.

"What about the twentieth in their pattern?" I then asked. "Decide in your group whether it would be a square or a column and why."

The group discussions were animated. I brought them back to attention after several hands were raised. I called on Jaimi.

"It would be a column," she said, "because the odd numbers are all squares and the even numbers are all columns." Again, there were nods of agreement.

I then gave a homework assignment. "You're to investigate your own growth patterns for homework," I said. I projected the directions for the assignment on the overhead:

Homework

1. What would your 20th pile look like? Describe in words.
2. How many tiles would you need to build your 20th pile? Explain how you know.

"As you're copying down these questions, I'll distribute the patterns you and your partner made for others to analyze," I said. "Each of you needs to make a sketch of your four drawings because I'm going to collect these patterns before you leave class. With the remaining time in class, talk with your partner about the problem. Tomorrow you'll compare your answers and reasons." They spent the rest of the time getting their homework assignment organized.

Day 5

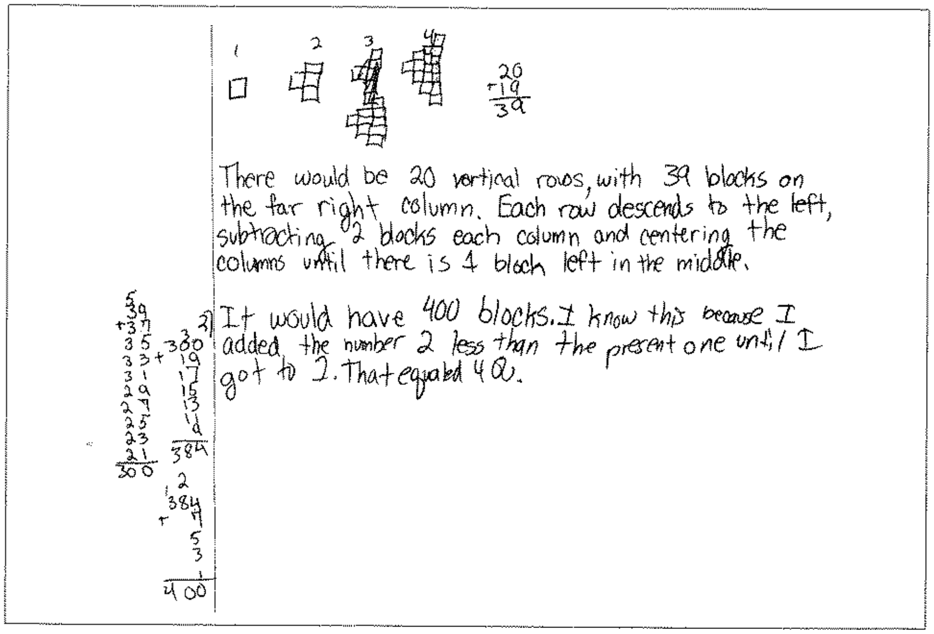
To begin class, I had pairs compare their homework as I circulated to check that they had done the work. I noticed that though a few students had difficulty with the assignment, most were able to describe how their twentieth pile would look and how many tiles were needed. Most had included diagrams in their descriptions.

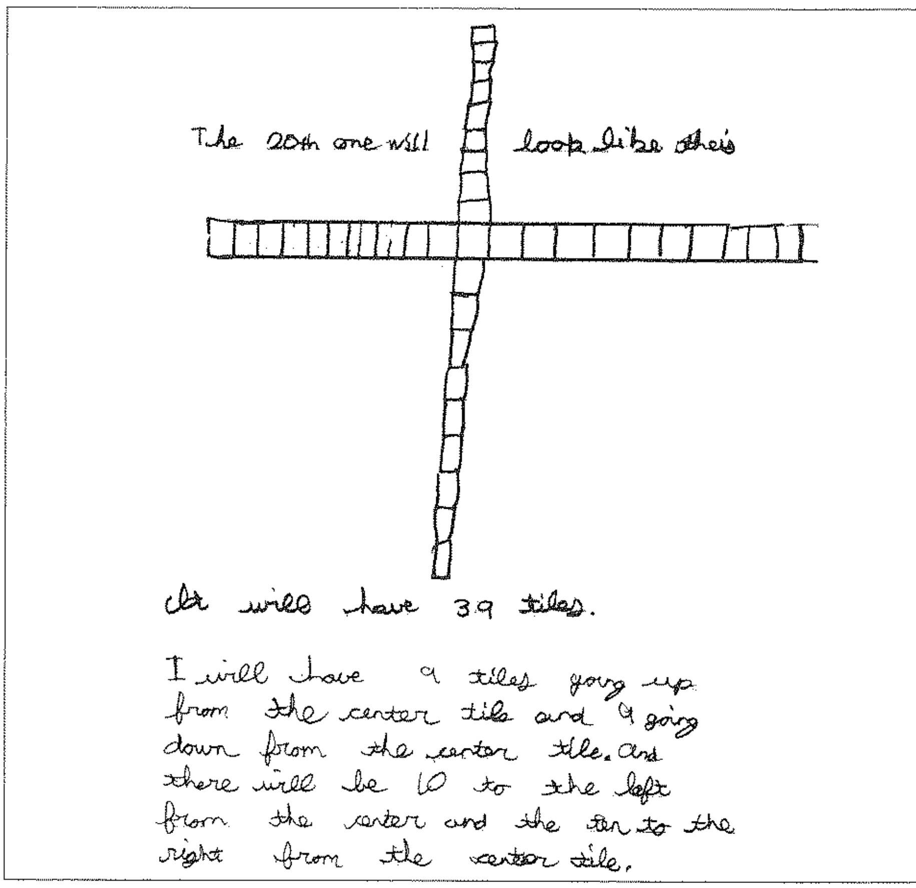
Roger wrote the following for his and Jessica's pattern: *The 20th one would have 20 tiles down the left and 20 tiles down the right. It would have 40 tiles because you just double it like number 1 had two tiles.*

Hal used a sketch to describe the twentieth one in his and Mark's pattern. He wrote: *The 20th one will look like this. It will have 39 tiles. I will have 9 tiles going up from the center tile and 9 going down from the center tile. And there will be 10 to the left from the center and ten to the right.*

Kelly and Cheryl had built a staircase pattern. Kelly wrote: *There would be 20 vertical rows, with 39 blocks on the far right column. Each row descends to the left, subtracting 2 blocks each column and centering the columns until there is 1 block left in the middle. It would have 400 blocks. I know this because I added the number 2 less than the present one until I got to 1. That equaled 400.*

My initial plan was to have pairs generalize and describe a method for figuring the number of tiles in any pile. I had even hoped to encourage them to write formulas. However, I decided to scratch this plan. I realized that most of their growth patterns were much more complicated than the $n + 1$ or $n + n - 1$ examples I had given. Also, they seemed to have spent enough time on their



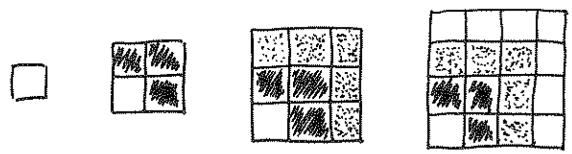


For homework, students described what their 20th pile would look like and how many tiles they would need to build it.

own patterns, so I decided to return to an investigation of the growth of squares and move them into several investigations about the growth of cubes. I collected their homework and called the class to attention.

"A few days ago, we looked at the pattern of squares," I began. "I want to return to this familiar pattern and again examine how squares grow."

I used overhead tiles to build four squares, using colors to accentuate how many were added to each square to get the next larger square.



I introduced the class to the story interpretation of growing squares presented in *Mouse and Elephant: Measuring Growth* (Shroyer and Fitzgerald 1986), one of the books in the Middle Grades Mathematics Project materials.

"Imagine that the smallest square is a square on its first birthday. It's a one-year-old square. The next one is a two-year-old square on its second birthday, and so on. Focus on how many tiles are added each year," I said, "and discuss in your groups how the squares grow." After giving them time to do this, I asked what they had noticed. Lots of hands went up.

"I'm interested in hearing all your ideas," I said. "Also, I want you to listen carefully to what others have to say."

"They grow in an L shape," Pam said. "You have to add a bigger L each time." Many nodded.

Brett reported next. "You go up by 2s," he said. I didn't understand what Brett meant. Richard interpreted for him.

"He means you add 2 more each time. Add 3, then add 5, then 7, and so on." Again, many nodded.

"How many tiles would be added to get from the nineteen-year-old square to the twenty-year-old square?" I asked. "Talk about this in your groups. You may need to use paper and pencil."

In a few minutes, I asked the class to come back to attention and called on Steven.

"It would be 39," he said. "We made a chart all the way to 20, and it came out to be 39." The class agreed that 39 was correct.

"Did anyone figure it a different way?" I asked.

Aurora hesitatingly raised her hand and tried to explain her complicated process. "Well, for the two-year-old," she said, "you added 3, and 3 is one more than 2, so 3 plus 2 equals 5 for the next one." I couldn't follow her reasoning. No one else could either. Aurora came up to the overhead and wrote the following:

3 years:	$3 - 1 = 2$
	$3 + 2 = 5$
4 years:	$4 - 1 = 3$
	$4 + 3 = 7$

She explained again. "You subtract 1 from the pile number and add that answer to the pile number and you get the number of tiles to add for the next square. So 20 would be 20 minus 1 equals 19 and 19 plus 20 equals 39."

Still, not too many others understood her reasoning. Jaimi, however, raised her hand. "Mine is almost the same as Aurora's, but not quite," she said. "Take any year. Take the year before it and add the two together. That will give you the number of tiles to add. For 20, you add 19 and 20, which is 39."

I then focused them on figuring the total number of squares. "You have different ideas for how many tiles to add each time," I said. "Now I'd like to have you look at the total number of tiles for each square. Do this for squares from one year old to ten years old."

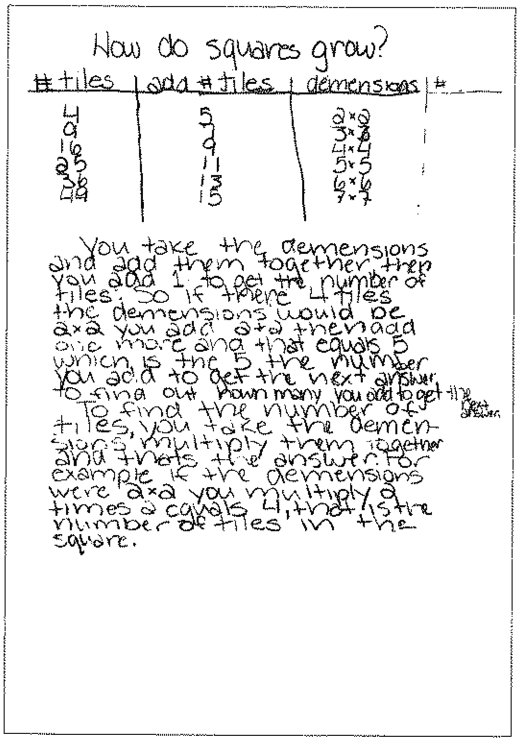
Some students starting writing on scratch paper. Some began to discuss in their groups. Others stared at the screen with thoughtful expressions. I had thought describing how squares grow would be immediately obvious to them, but it wasn't. After a few minutes, I called the class to attention. Kelly explained what she had discovered.

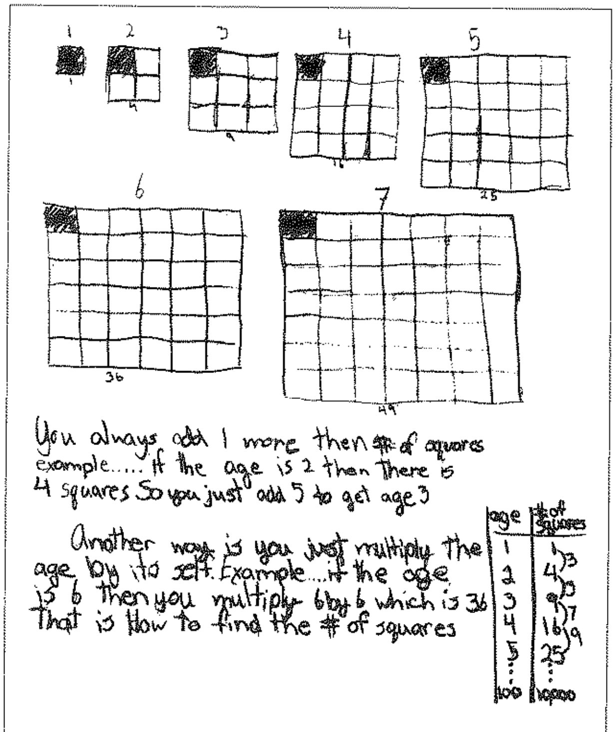
"I found a pattern," she said, "and I can tell how many tiles no matter how old the square is. For 1, it's 1; 2 is 4; 3 is 9." I recorded as she reported. "1 times 1 is 1," she elaborated. "2 times 2 is 4, 3 times 3 is 9, and so on. So for the ten-year-old, it would be 100 because 10 times 10 is 100." About half the class nodded. No one had a different way to report.

I then gave the students an assignment to do individually. "Describe all you can about how squares grow," I said. "Show your data, include drawings, and give convincing arguments for what you write." The class settled into work.

Greg drew eight squares of increasing sizes. He wrote: *It gets gradually bigger and it is amazing how fast it grows. Look how big the 8 is compared to 1.* $1 \times 1 = 1$, $2 \times 2 = 4$, $3 \times 3 = 9$, *etc. Multiply the age by itself and that gets the answer.*

Conroe wrote: *You always add 1 more than the # of squares. Example . . . if the age is 2 then there is 4 squares, so you just add 5 to get age 3. Another way is you just multiply the age by itself. Example . . . if the age is 6 then you multiply 6 by 6 which is 36. That is how to find the # of squares.*





Students were asked to describe all they could about how squares grow.

Kimberly wrote: *You take the dimensions and add them together, then you add 1 to get the number of tiles to add to get the next answer. So if there are 4 tiles you add 2 and 2 and then one more and that equals 5 which you add to 4 to get the next answer. Or you take the dimensions and multiply them together. For example, you 2 times 2 equals 4. That is the number of tiles in the square.*

At the end of class, I told them that tomorrow we'd investigate how cubes grow.

Day 6

To prepare for the next day, I assembled a small plastic bag for each group with about 100 two-centimeter interlocking cubes, the kind that can be snapped together to make three-dimensional shapes. Also, I listed these terms on an overhead transparency:

faces

edges

vertices

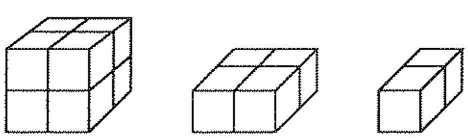
To begin, I left the plastic bags on the counter and gave each group just 1 cube. I held up a large cube for the class and asked, "Can anyone explain what *faces* or *edges* or *vertices* of a cube are?"

The only word they knew for sure was *vertices*. (We had studied angles earlier in the year.) I told them what faces and edges were.

"Discuss in your groups how many faces, edges, and vertices a cube has," I said. All of the groups counted 6 faces and 8 vertices correctly, but more than half of them counted only 8 of the cube's 12 edges. I've found this to be a common mistake.

I then held up one of the two-centimeter cubes. "This is the size of a one-year-old cube," I said. "Have one person get a bag of cubes for your group and then build what you think is a two-year-old cube."

They got involved enthusiastically. There were three different solutions. Three groups correctly built a 2-by-2-by-2 cube. Two groups snapped together 4 cubes, and two groups used only 2 cubes.



I called the class to attention and held up the large cube and a box of tissues. "Both of these have 6 faces, 8 vertices, and 12 edges," I said, "but the box of tissues isn't a cube. What makes a cube a cube?"

Steven raised his hand. "They're all squares," he said. "I mean, every side is a square."

"Why isn't the tissue box a cube?" I asked.

Rena answered. "Some of the sides are rectangles," she said.

"Yes," I said, "in a cube, all the faces are squares." I modeled the correct terminology. "Some of the structures you built to represent a two-year-old cube aren't cubes. Try again."

After a moment, all of the groups had correctly built a 2-by-2-by-2 cube. I then explained the investigation I wanted the students to try.

"As a group," I said, "build a two-year-old, a three-year-old, and a four-year-old cube. For each, answer two questions. How many one-year-old cubes does it take to build it? And how many cubes are there that aren't visible from the outside?" I wrote the questions on the overhead.

"I don't get it," Mark said.

"With a two-year-old cube," I explained, "you can see all 8 cubes you used to build it. But with larger cubes, some will be inside."

"I know," Elizabeth called out, "there's one inside the three-year-old cube." She already had assembled a three-year-old cube, and she showed it to the class. The class was intrigued. The idea of cubes not being visible seemed to spark their curiosity.

"After you've investigated the two-year-old, three-year-old, and four-year-old cubes," I said, "look for patterns and see what you can predict for larger cubes."

I circulated and watched the groups work for about fifteen minutes. Jaimi's group was upset with her. As the recorder, she was writing numbers without consulting the others. I saw that all her predictions were wrong for the cubes that weren't visible. I talked with the group about the importance of using the materials to verify their ideas and the importance of the recorder's recording only what the entire group agreed on. They worked more effectively after that.

In another group, Rodney and Sharlene, who were both usually reluctant math students, got very interested. They were systematic and helpful in their group's investigation.

David had built a four-year-old cube with a two-year-old cube inside that could be removed. Pam's group had made a hollow six-year-old cube. I gathered some of the students to show how cubes of some sizes fit inside others. This sparked a new direction in the class, and other groups began to experiment.

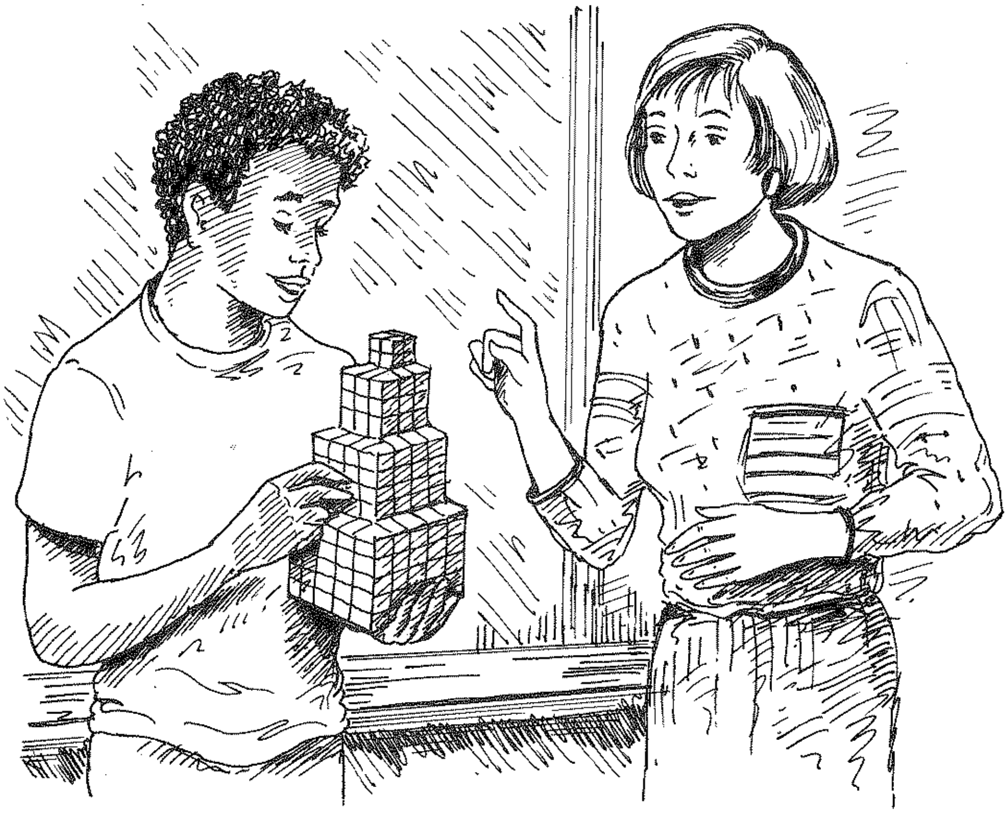
Cheryl, Rena, Natalie, and Huy came up with a generalization. They wrote: *In a 4 year old cube you can fit a 2 year old cube inside, in a 5 year old cube you can fit a 3 year so you minus 2 to get the one you can fit into.*

Aurora, Tiambe, and Elizabeth came to a similar conclusion. They wrote: *To get the no. of cubes not visible you minus 2 from the age then figure out the no. of 1-year old cubes in that age.*

Three other groups had figured out how many invisible cubes there were for three-year-old, four-year-old, and five-year-old cubes but hadn't related the numbers to dimensions of cubes and didn't come to generalizations. Two groups were working more slowly and were concentrating on the number of cubes needed to build each one. I encouraged them to think about the invisible cubes as well.

Rodney came up to me with a two-, three-, four-, and five-year-old cube, one stacked on top of the other. "It's just like the squares," he said excitedly. "There's an L around them."

It was now near the end of class, so I had them get organized and ready to leave. "Don't break apart your cubes," I said. "Put them in the bags intact and put in a slip of paper with your group's names on it. You'll continue tomorrow." I collected their work. It had been an exciting class.



Rodney stacked two, three, four, and five-year-old cubes to show his discovery about how they grow.

Days 7 and 8

I began class by reading aloud the generalization that Cheryl, Rena, Natalie, and Huy had written. "Discuss in your groups why this is true," I said to the class. "Use the cubes to prove it. If you're stuck, raise your hand, and Cheryl, Rena, Natalie, or Huy will come and help." I knew that not every student was going to grasp this generalization now. Some would need more experience before they would be able to understand it. But I wanted to begin class by getting them back into thinking about the cube patterns they had been investigating.

I also directed a comment at the group that had written the statement. "Talk about how you could convince someone that what you wrote is true," I said.

I read the statement once more and let the groups discuss it for a few minutes. I then called the class back to attention and introduced another investigation about cubes.

"Today you'll begin to explore a problem about painted cubes," I said. I held up a 2-by-2-by-2 cube.

"Imagine that I dip this cube in a bucket of paint," I said, "a bucket of magic paint that dries instantly. Then I remove the cube from the bucket and break it apart into one-year-old cubes. How much of each one-year-old cube is painted?"

3 year old = 27 one year old cubes
2 years old = 8 one year old cubes
4 years old = 64 one year old cubes

* In the three year old cube there is 1 cube you can not see.
In the four year old cube there are 8 cubes you can't see.

* It takes 15,625 cubes to make a 25 year old cube.

In a 4 year old cube you can fit a 2 year old cube inside, in a 5 year old cube you can fit a 3 year so you minus 2 to get the one you can fit into.

Cheryl, Rena, Natalie, and Huy wrote about their investigation of cubes that grow.

Most of the students immediately saw that three faces of each cube would have paint on them. Richard explained.

"Each cube was on a corner," he said, "and three sides were showing." This seemed obvious to many of the students.

"OK," I continued, "suppose I did the same with a 3-by-3-by-3 cube. How much of each cube would have paint on it? Talk about this in your groups."

The groups all scrambled for three-year-old cubes. Their discussions were animated. I overheard a variety of comments.

"Look, there's one inside that doesn't get painted at all."

"The corner ones all have three sides painted."

"How many corners are there?"

"The rest only get painted on one face."

"No, here's one that would get two faces painted."

"Is there only one without paint?"

"There's nine on each face, so I think you do 6 times 9."

I called them back to attention and asked what they had discovered. They reported the kinds of statements I had overheard.

"Were there any cubes with more than three faces painted?" I asked. Several students answered no.

"Could there be if the cube were larger?" I asked. "Talk about this in your groups."

In a moment, I called them back to attention and called on Pam. "The ones in the corners get the most paint," she said, "and they only have three sides showing. There's no way to get more painted."

"So you found cubes with three, two, one, and zero painted faces," I said. There were nods.

I drew a chart on the overhead projector and had the students help me fill it in:

Age	0 faces	1 face	2 faces	3 faces
2-yr-old	0	0	0	8
3-yr-old	1	6	12	8

I then described the investigation I wanted the students to do. "In your groups, investigate what happens to larger cubes when they're dipped in the bucket of magic paint. Make a chart like the one I've shown and continue recording. Look for patterns."

"How high do we go?" Pedro asked.

"I suggest you do at least four-year-old and five-year-old cubes," I answered. "Then see if you have enough information to predict what would happen for a six-year-old cube. Write statements about any patterns you find."

The students were eager to get started. They worked for the rest of the period and continued the next day. Interest stayed high. As I circulated, I encouraged students to write generalizations about what they noticed.

Aurora wrote the following for her group: *To predict how many are hidden minus 2 from the age and bring that number to the 3rd power. To get how many have 1 face painted subtract 2 from the age and multiply that no. by itself then the answer by 6. To get the no. of how may cubes have 2 faces painted subtract 2 from the age then multiply that no. by 12. All cubes have 8 separate cubes with 3 faces painted.*

Richard wrote for his group: *3 faces painted is always 8 because there are 8 corners. The cubes that are needed is age^3. 2 faces painted is always age minus 2 × 12 because on one edge there is 2 less than the age because you don't count the corners and there is 12 edges. 1 face is always age minus 2 × age minus 2 × 6 like the one above because its a square then times 6 because there is 6 faces. 0 faces is the number of cubes needed of a cube with a two age difference.*

I had these two groups compare their statements to decide whether they had come to the same conclusions.

Tami's group came up with different sorts of statements. They looked at the patterns of numbers in the charts but not at how they were generated. They wrote: *For cubes needed the last number goes in a pattern odd, even, odd, even, odd, even. For 1 face painted they are multiples of six. For 2 faces painted just add 12. For 3 faces painted they are all even numbers.*

Groups that finished early compared their statements. For homework, I asked students to write what they knew about cubes. Mark asked how much they needed to write.

"About one page," I answered.

Reading students' writing gives me information about what they experienced and helps me be more sensitive to individuals' needs. Most students revealed that they had a basic sense of what cubes were.

Age	0 faces	1	2	3
3 year old	1	6	12	8
4 year old	8	24	24	8
5 year old	27	54	36	8
6 year old	64	96	48	8
7 year old	125	150	60	8
10 year old	216	384	96	8

Prediction Pattern

To predict how many are hidden minus 2 from the age and bring that number to the 3rd power. To get how many have 1 face painted subtract 2 from the age and multiply that no. by itself then the answer by 6. To get the no. of how many cubes have 2 faces painted subtract 2 from the age then multiply that no. by 12. ~~To get how many cubes~~ All cubes have 8 separate cubes with 3 faces painted.

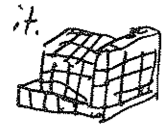
3 faces painted is always 8 because there are 8 corners.

The Cubes that are needed is age³ because l × w × h = volume.

2 faces painted is always age − 2 × 12 because on one edge there is 2 less than the age because you don't count the corners and there is 12 edges.

1 faces is always age − 2 × ? like the one above (no corners) × 2 because its a square then times 6 because there is 6 ~~sides~~ faces.

0 faces is the number of cubes needed of a cube with a two age difference. The big one covers it.



Working in groups of four, students presented their work about the painted faces of cubes.

Elizabeth's paper was typical of many students' papers. She listed what she knew:

Cubes have eight corners always.

Cubes have six sides always.

Cubes have twelve edges.

Cubes can be made into bigger cubes.

A ten-year old cube has 1,000 cubes in it.

When building cubes to figure out how many have two faces painted you just add twelve starting with a two year old cube and zero cubes with two faces painted.

Some students wrote paragraphs. Often, they provided information that hadn't been included in these activities. Pam, for example, wrote: *I know that every cube has to have six even faces, and eight vertices. They also have to have twelve edges. I know that a cube is not a square because a square is two demensional and a cube is three demensional, also a square has area and a cube has volume. I know that if you throw a cube up in the air it has a 16½% chance that one face will show up.*

Art expressed his lack of success and confidence. He wrote: *Well, I've been struggling with this particular subject. I know they have 6 faces, 8 corners, and 12 edges, and all that but I just have a lot of trouble. I was thinking about coming after school for help. I'm sorry I can't tell you more. I know more things about it but I don't know how to put them in words.*

I had students share what they wrote with the others in their groups. I talked with Art and the other students who had said they were having trouble. Though we took a break from focusing on patterns after these experiences, the thinking permeated all that we did afterward. Students naturally looked for patterns in whatever math they were exploring. That was a worthwhile payoff.

Final Thoughts

Teaching five classes a day makes it hard to get to know students in great depth. It's especially hard when I teach from the front of the room and only hear from those confident enough to offer their thoughts in a class discussion. However, in lessons such as described in this chapter, when students are actively investigating problems, working with concrete materials, and interacting with one another, I have more opportunity to observe them and learn about their thinking processes. Also, the students are more invested in their own learning. They're involved, they communicate their thoughts, and they seem to enjoy math class. Though getting organized for these classes was demanding, the results justified the effort.