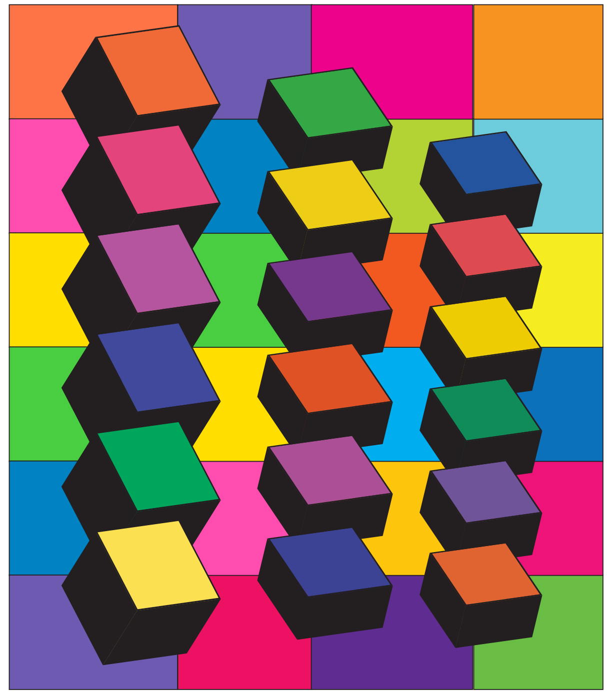
Student Explorations in Mathematics

The Algebra of Patterns



One goal of this activity is to introduce students to the idea that a function can be thought of as an algebraic description of a mathematical pattern. Students will explore the domain and range of a function and graphs of functions with continuous and discrete domains.

Students begin this activity using tiles to build figures that represent a growth pattern. The investigation will be enhanced with the use of a manipulative, but students can also draw the models. Students identify and describe a pattern in the sequence. The description takes two forms: a verbal description and an algebraic expression. Writing the verbal description may be challenging, and the teacher should be prepared to spend some time on the task, allowing students to develop their own ideas and use their own language. Multiple approaches and descriptions may be valid. Similarly, when students write algebraic expressions to describe the pattern, they need not put their answer into simplest form immediately. Depending on the student and class, some may do this automatically. Honor all solutions, simplified and otherwise, and seize the opportunity to ask students if the solutions are equivalent.

After individual students have written the algebraic expressions, the class can work together to match the expressions to the correct verbal model. This allows students to see connections between algebraic symbols and the physical model. Students then repeat the exercise with different sequences. Starting values and rates of change are similar to those of the original sequence. Depending on student background, the teacher can introduce or support the ideas of slope and *y*-intercept, which can lead into a discussion of linear functions.

TEACHER NOTES

Suggested answers are in **red**. Instructional notes are in **blue** and are preceded by the hand icon. These two features do not appear in the student edition.

Classroom materials

- Colored tiles or other manipulatives for modeling sequence arrangements

Supplemental files

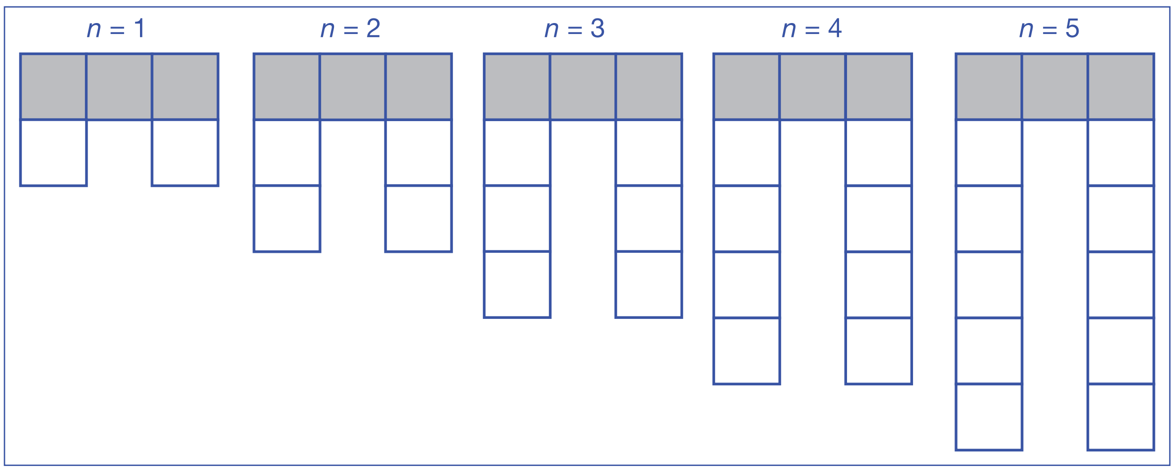
- Student Edition

Extended Teacher Notes for "The Algebra of Patterns"

This page does not appear in the Student Version.

Question 5

Approach 1: The student sees the three tiles across the top horizontal row as a constant, with growing vertical columns. The first repeated-addition column in the table counts the pairs of tiles added at each stage. The second repeated addition counts each row of vertical tiles. Use one or the other, based on student input.



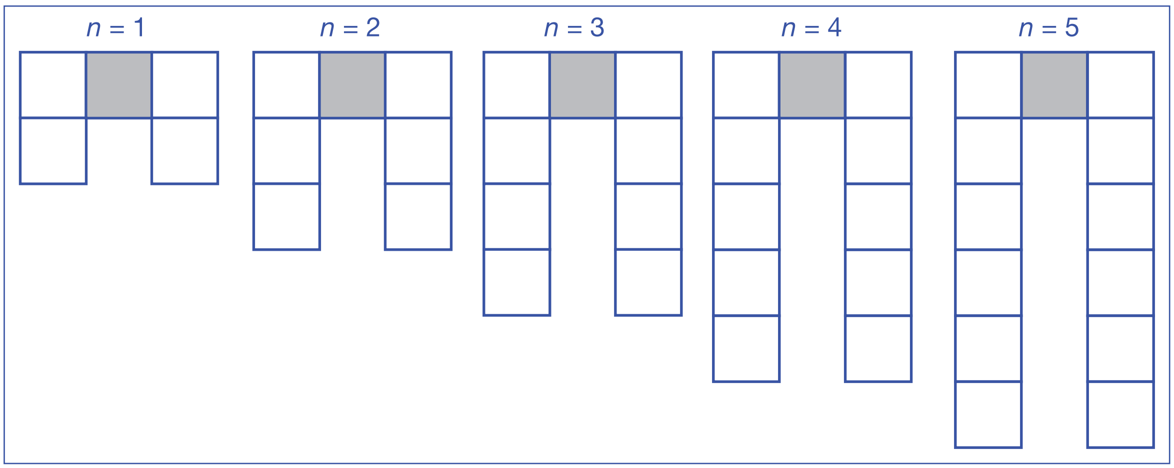
n	Repeated Addition	Alternate Repeated Addition	Expression	Total Tiles
1	3 + 2	3 + 1 + 1	3 + 2(1)	5
2	3 + 2 + 2	3 + 2 + 2	3 + 2(2)	7
3	3 + 2 + 2 + 2	3 + 3 + 3	3 + 2(3)	9
4	3 + 2 + 2 + 2+ 2	3 + 4 + 4	3 + 2(4)	11
5	3 + 2 + 2 + 2 + 2 + 2	3 + 5 + 5	3 + 2(5)	13
n			$3 + 2(n)$	$2n + 3$

Extended Teacher Notes for "The Algebra of Patterns" (cont.)

This page does not appear in the Student Version.

Question 5

Approach 2: The single, shaded square is the constant, with longer vertical columns. Again, two different possible counting approaches are shown. Students would use only one.



n	Repeated Addition	Alternate Repeated Addition	Expression	Total Tiles
1	1 + 2 + 2	1 + 2 + 2	1 + 2(2)	5
2	1 + 2 + 2 + 2	1 + 3 + 3	1 + 2(3)	7
3	1 + 2 + 2 + 2 + 2	1 + 4 + 4	1 + 2(4)	9
4	1 + 2 + 2 + 2 + 2 + 2	1 + 5 + 5	1 + 2(5)	11
5	1 + 2 + 2 + 2 + 2 + 2 + 2	1 + 6 + 6	1 + 2(6)	13
n			$1 + 2(n + 1)$	$2n + 3$

Extended Teacher Notes for “The Algebra of Patterns” (cont.)

This page does not appear in the Student Version.

Question 9

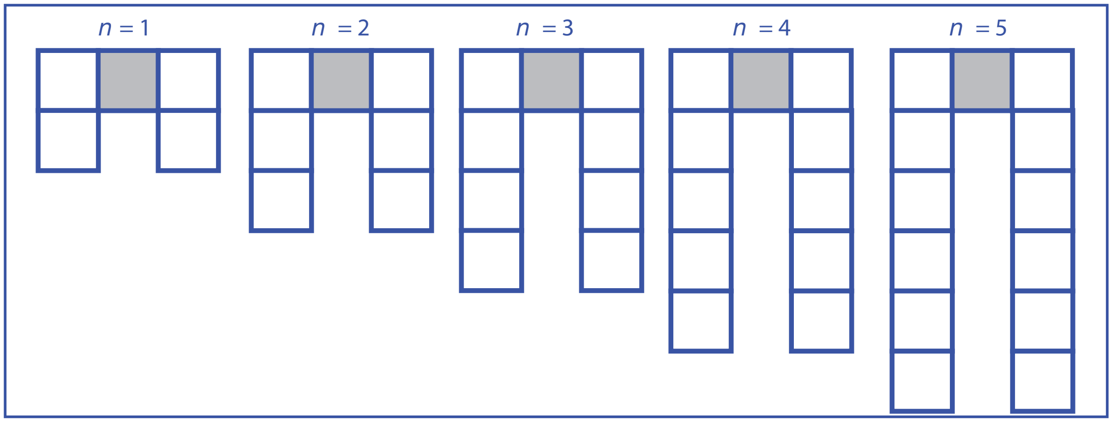


Figure T-3A

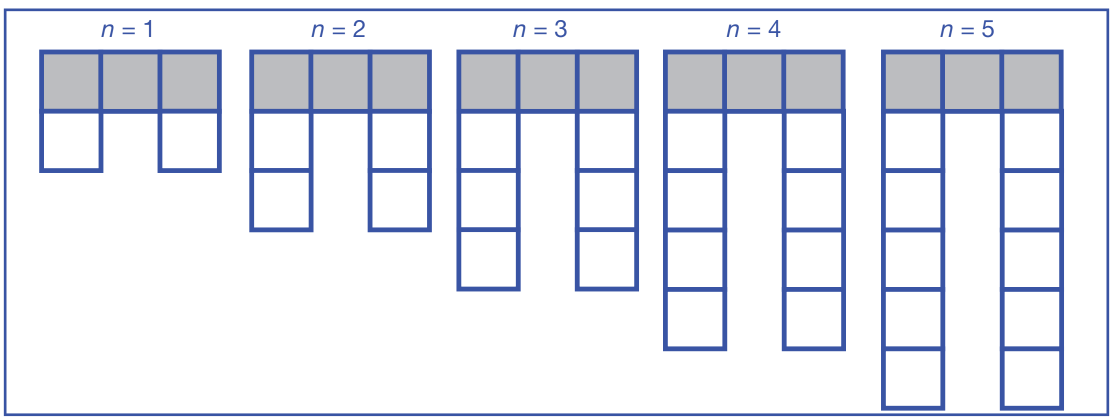


Figure T-3B

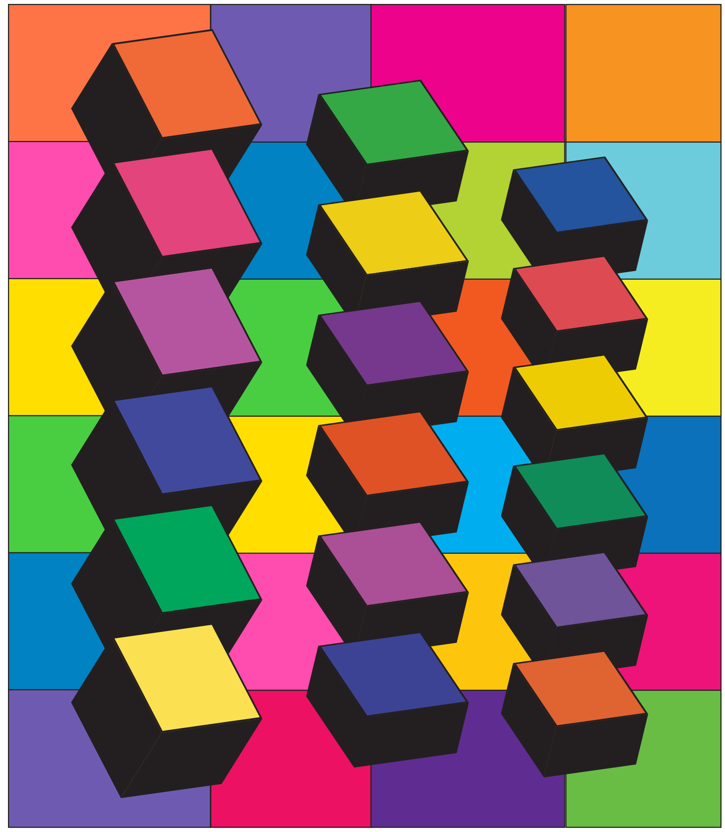
Possible answers:

- $2(n + 1) + 1$: The two vertical columns each have one more tile than the stage number, and one tile is always in the middle.
- $3 + 2n$: Three tiles on the top horizontal row stay constant; the rest of the tiles are always two times the stage number.
- $3(n + 1) – n$: Think of the figure as a rectangle with some tiles missing; the rectangle is three tiles across and $n + 1$ tiles down, with n tiles missing.
- $2(3) + 2(n + 1) – 4 – 1$: Imagine adding a tile to the middle of the bottom row to make a complete rectangle that has two rows of three tiles each and two columns of $n + 1$ tiles each. We have counted each corner tile twice, so we subtract the four tiles counted twice as well as the tile we added.

Ask students to explain their choices. Encourage both student discussion and alternate expressions for the same verbal description. Some counting methods will involve regrouping tiles; others use area and perimeter models or adding or removing tiles. A variety of approaches is more important than a single correct answer. Showing that all the expressions are equivalent is an important teaching moment.

Student Explorations in Mathematics

The Algebra of Patterns



Mathematicians often use functions to describe patterns. In this activity, you will build figures that follow a pattern and describe how the figures change. By the end of the unit, you will have a clearer understanding of what a function is and how to use one to describe a pattern.

In a sequence, the first figure will be referred to as the stage 1 figure, the second will be stage 2, and so on. So, being asked for the stage 15 figure means you would create the fifteenth figure in the pattern. The variable *n* represents the stage number.

The first question asks students to create the fourth figure in a pattern. You may be surprised at some of the patterns that students will find. More than one possible answer exists. Allow students to defend their answers, and ask the class to agree or disagree with each. As the activity progresses, students will be asked to narrow their investigation to a certain pattern.

1. A student used tiles to build the sequence below. From one stage to the next, what changes? Following the pattern, use tiles to build stage 4. How many tiles did you use?

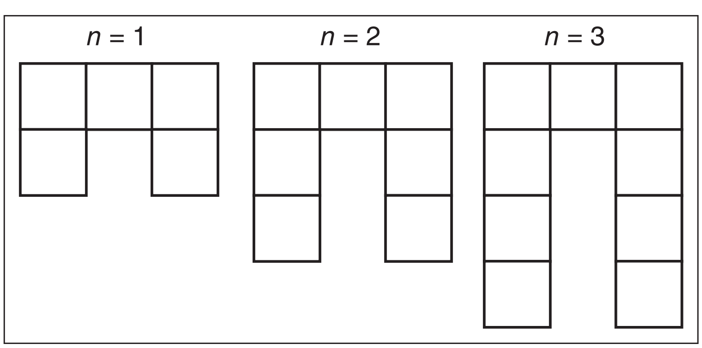


Figure 1

The number of tiles, perimeter, area, and number of lines are all examples of things that change. One possible stage 4, with 11 tiles, is shown below.

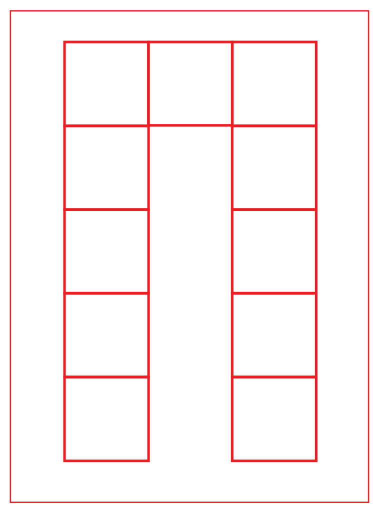


Figure 1 Answer

Welcome any and all observations, but do not lead students into any particular characteristics of the sequence. Some students may suggest that the set of three figures will keep repeating, so the fourth figure will be the same as the first figure in the given set. However, if a student determines that the fourth figure looks like the one given in the answer key, he or she will have recognized a growing pattern in the set of figures. Both results are valid. Let students know that this activity will focus on growing patterns and how to describe them.

2. Use your tiles to build the stage 9 figure of this growing pattern. How many tiles did you use? Explain how you know this is the correct number of tiles.

21 tiles

✍ Students will produce a variety of explanations for the stage 9 sequence. Some may use tiles to build the stage and may count how many tiles they use. Others may create a chart. Still others may see a pattern of starting with five tiles and adding two tiles at each stage. Those students who find the ninth figure by referring to the eighth, which refers to the seventh, and so on, are using what mathematicians call a *recursive rule*.

Some students may see that doubling the stage number and adding three gives the total. Some may see that adding one to the stage number, doubling that, and then adding one more gives the total. These last two patterns are called *explicit rules* because the total number of tiles is determined by only the stage number not by the change from the previous figure. Welcome all descriptions, and ask students if they agree that the descriptions are accurate.

3. How many tiles would be in the stage 100 figure? Use words to describe how you know.

203

✍ At this point, students should notice that using a recursive rule or trying to build the figure is cumbersome. If you notice students attempting to build the figure or use a recursive rule, allow productive struggle to drive a conversation about finding a more efficient method to count the tiles. So that each student can arrive at his or her own conclusion, consider having students wait to share their solutions. Challenge them to find more than one solution. The sketches on page iv show two of the many ways students may see the pattern. **Figure T-3A** on page iv shows two growing columns with an extra tile in the middle. **Figure T-3B** on page iv shows one row of three tiles with two growing columns underneath.

4. How many tiles would be in the stage n figure? Write your description in words. This is called a *verbal description of the sequence*.

Possible verbal descriptions:

- The three tiles in the top horizontal row stay constant; the remaining number of tiles is always two times the stage number.
- The two vertical columns each have one more tile than the stage number, and there is always one tile in the middle.
- Think of the figure as a rectangle with some tiles missing: The rectangle has dimensions three tiles across and $n + 1$ tiles down, with n tiles missing.
- Imagine adding a tile to the middle of the bottom row to make a complete rectangle; that rectangle has two rows of three tiles each and two columns of $n + 1$ tiles each, and we have counted each corner tile twice, so subtract the four tiles counted twice and the tile we added.
- Other verbal descriptions are possible.

✍ Many answers are possible. Encourage different approaches. If a group completes this task quickly, challenge the group members to find other methods. Have students display their answers in the classroom. Also display descriptions that students do not generate. You may even want to display an incorrect description as a challenge. Have students discuss which descriptions they agree and disagree with. Answering this question may be a difficult leap for some students. Some people define algebra as the generalization of arithmetic, and students may not be ready to generalize from the specific to the abstract. Some can see and do this without formal training. Others need leading questions and guidance. If a student develops an elegant or intriguing description, have him or her email their solution to **sem@nctm.org**.

5. Choose one of the verbal descriptions generated by your class. Use that model to write an algebraic expression for the number of tiles in the stage n figure.

Possible expressions (among others):

- $3 + 2n$
- $3(n + 1) - n$
- $2(n + 1) + 1$
- $2(3) + 2(n + 1) - 4 - 1$

✍ Encourage students to choose different verbal descriptions so that each model proposed has a corresponding algebraic expression. Have them display the functions but not next to the corresponding verbal descriptions. See pp. ii and iii for extended teacher notes and some suggestions of how students can be guided to the algebraic generalization through sketches and tables.

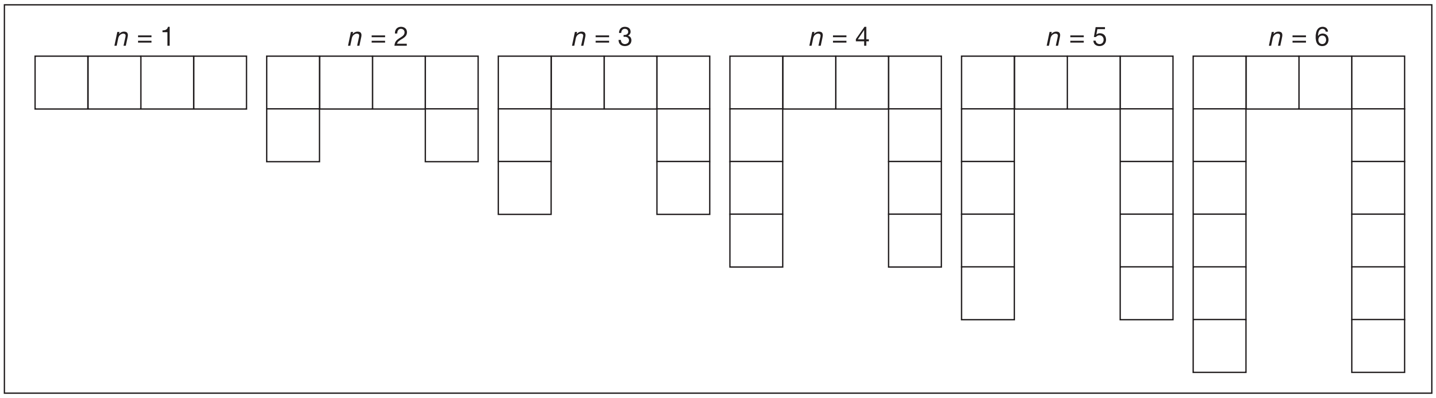


Figure 2

6. Match each expression displayed in the classroom to the correct verbal description.

Ask students to explain their choices. Encourage both student discussion and alternate expressions for the same verbal description. Some counting methods will involve regrouping tiles; others use area and perimeter models or adding or removing tiles. A variety of approaches is more important than a single correct answer. Showing that all the expressions are equivalent is an important teaching moment.

Figure 2 above shows the first six stages of a growing sequence.

7. Describe the number of tiles in the stage n figure with both a verbal description and an algebraic expression.

One possible answer: Two vertical rows of n tiles each with two additional tiles in between; $2n + 2$

8. How is this set of figures similar to the first set you worked with? How is it different?

The sequences have the same recursive change, adding two tiles every time, but a different starting value. The expressions both contain a term of $2n$, but they have different constants.

Figure 3 below shows the first five stages of a growing sequence.

9. a. Describe the number of tiles in the stage n figure with a verbal and an algebraic expression.

$2n + 3$, or any equivalent expression

   b. Compare and contrast this sequence with the first sequence of this activity.

The sequences can be represented by the same algebraic expression, even though the physical models are different.

This is a rich opportunity to talk with students about rates of change, slope, and intercepts. Encourage a class discussion that relates the descriptions they are writing to their previous work with linear functions.

10. Create a sequence of diagrams that can be described by the expression $4n + 3$. Which stage has 47 tiles?

The eleventh figure will have forty-seven tiles.

Mathematicians use functions to describe how patterns change. Using physical models, verbal descriptions, and mathematical symbols, you have described how the patterns in this activity change. Two other ways to display the information are in tables and graphs. Functions can be represented by equations, tables, words, or graphs.

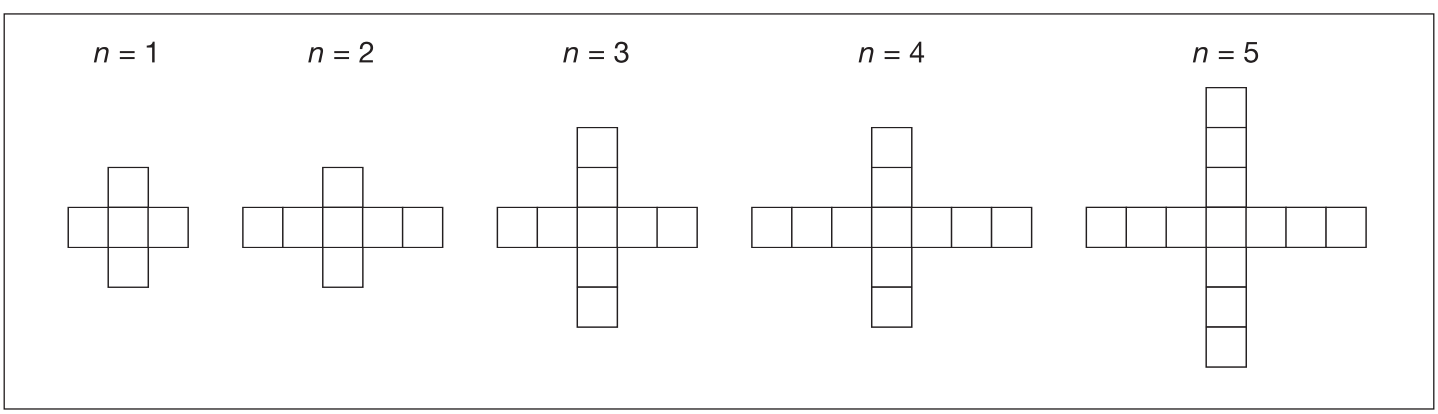


Figure 3

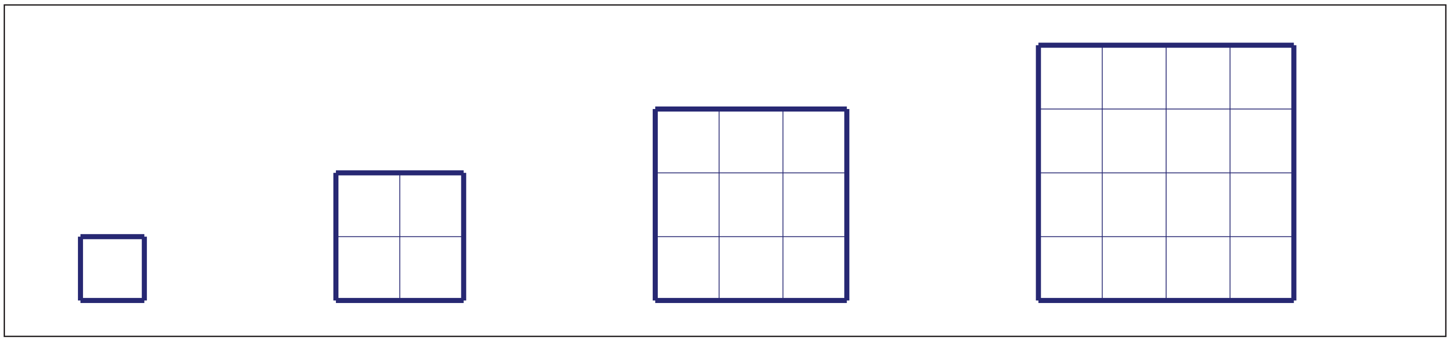


Figure 4

11. Consider the sequence consisting of the squares of the counting numbers (see **fig. 4** above). Complete **table 1** below, where the input, *n*, is the stage number and the output is the number of tiles in the *n*th drawing.

The *domain* of a function is a list of all input values (the first column of **table 1**). The *range* is a list of all of the output values.

12. What kind of numbers make up the domain of the function in question 11? What about the range?

The domain is the set of all positive integers (natural numbers). The range is the set of perfect squares, where we use the term *perfect square* to represent the square of an integer.

Point out to students that the table represents only some of the function values, which is why the domain and range include more values. This section is heavy on vocabulary; individual teachers should decide on the right focus for their students. In addition to the commonly addressed concepts of domain and range, an opportunity exists to focus on the differences between number sets. Some students may say that the domain is "all numbers," not distinguishing integers from nonintegers, or even positive from negative.

Table 1

n	tiles
1	1
2	4
3	9
4	16
5	25
6	36
7	49

13. Plot the points from **table 1** onto a set of coordinate axes (see **fig. 5**). Use the domain for the horizontal axis and the range for the vertical axis. Should the points be connected?

Some students will say not to connect the points because input values (stage numbers) between integers do not make sense. Others may say to connect the points because the input is the length of the side of the square, which can be any positive number.

Possible answers show a difference between a discrete function and a continuous function. If we think of each input value as a stage number, only integral values greater than zero make sense (there is no stage 3.5). Because values between data points are not included, the points are not connected and the resulting function is a set of discrete points. When

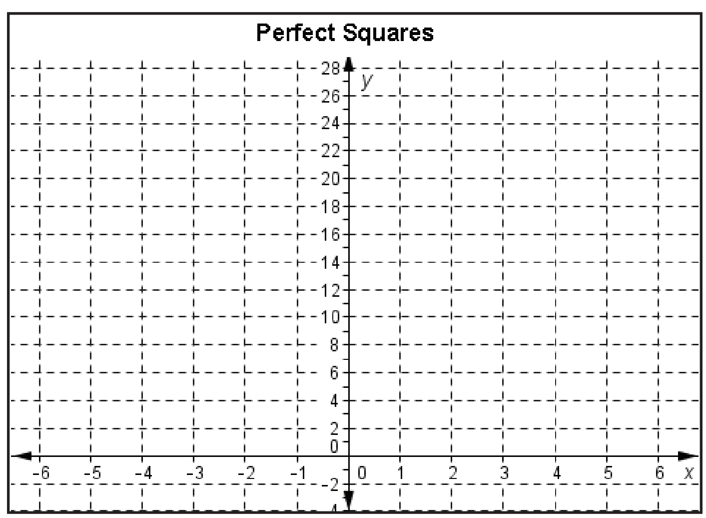


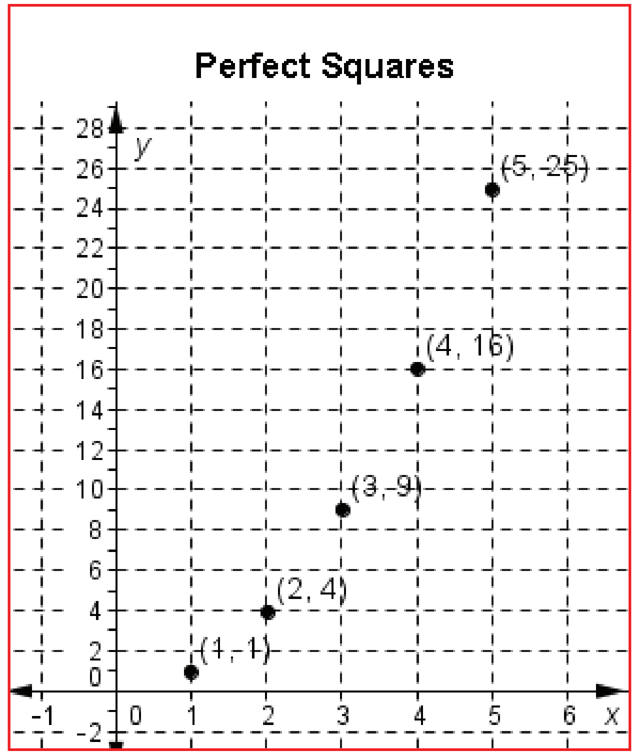
Figure 5

all values between the data points are included in the domain, the points are connected and the function is continuous. The exploration of ideas is more important than agreeing on an answer. Encourage this conversation, extending the focus to number sets. Note that if you assume a continuous function, inputs and outputs can be irrational.

14. We will assume the given function is continuous, so we do connect the data points. Extend the graph to include negative inputs. What are the domain and range of the new function you have drawn?

The domain is the set of all real numbers. The range is the set of all real numbers greater than or equal to zero.

The teacher can determine how much notation students are ready to use. The notation for the domain and range is $x \in \mathbb{R}$ and $y \in \mathbb{R}$ and $y \geq 0$.



Figures 13–14 Answer

Can you . . .

- create a sequence of diagrams that can be modeled by the expression $5n - 2$?
- create a sequence of diagrams that can be modeled by the expression $n^2 + 4$?
- create a sequence of diagrams that can be modeled by the expression $(n + 2)^2$?
- create a sequence of diagrams that can be modeled by the expression $\sqrt{n} + 3$?

Did you know . . .

- that three common types of functions are linear, quadratic, and exponential? Find out how each of these functions is defined.

Sources

The Math Learning Center. 2005–2013. Bridges in Mathematics Curriculum. 2nd ed. Salem, Oregon: The Math Learning Center. http://www.mathlearningcenter.org/

Murdock, Jerald, Ellen Kamischke, and Eric Kamischke. 2001. *Discovering Advanced Algebra: An Investigative Approach*, vol.1. Peabody, MA: Key Curriculum Press. http://books.google.com/books/about/Discovering_Advanced_Algebra.html?id=_JGqGgAACAAJ

Student Explorations in Mathematics is published electronically by the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 20191-1502. The five issues per year appear in September, November, January, March, and May. Pages may be reproduced for classroom use without permission.

NCTM® NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS