UNIT 9

Using Area Models to Understand Polynomials

Lesson Plan

This lesson plan explores a concrete way for students to conceptualize multiplication that ultimately leads to a deeper understanding of abstract algebraic topics, including multiplication and factorization of polynomials. We hope that this approach will allow both teachers and students to come away with a better sense of how multiplication of polynomials is connected to multiplication of integers. We should dispel the myth that math is a discrete set of topics. Ideally, we will see math more as "an interconnected body of ideas" (Swan, 2005).

In this lesson, we start with intuitive images of arrays, move to concrete representations of area with manipulatives and graph paper, and continue in scaffolded steps towards an abstraction of the area model of multiplication, which we will use to multiply polynomials.

Check out Patricia Helmuth's CollectEdNY.org post on how she uses online area model manipulatives from the National Library of Virtual Manipulatives site to help students practice multiplication of whole numbers and fractions (http://www.collectedny.org/2015/02/deepen-conceptual-understanding-in-math-with-virtual-manipulatives/)

This lesson is made up of the following sections, which should be followed in order:

1. **Using Arrays to Explore Multiplication**
2. **A Measured Area Model**
3. **The Distributive Property**
4. **An Abstract Area Model**
5. **Applying an Area Model to the Multiplication of Polynomials**

We recommend using these activities over a series of classes. In addition to developing fluency with multiplication of integers, fractions, percents, monomials, and binomials, the area model has the added usefulness of helping students understand area and perimeter. This model can create a shared visual language to refer back to when students struggle with multiplication in different contexts.

FOOTNOTE: We would like to thank the following educators for their inspiring workshops at the Commission on Adult Basic Education (COABE) 2015 conference: Amy Vickers, whose presentation on Rectangles as Problem Solving Tools greatly informed our understanding of the use of area models for teaching all forms of multiplication, including polynomials, as well as Lynda Ginsburg and Patricia Helmuth, whose workshop on quadratics and visual models demonstrated visual ways to contextualize factoring of quadratics. Amy's fantastic presentation, *Rectangles as Problem-Solving Tools: Use Area Models to Teach Math Concepts at All Levels,* can be found at http://adultedresource.coabe.org.

OBJECTIVES

Students will use area models to…

- ✔ Understand the connection between multiplication of integers and multiplication of polynomials.
- ✔ Understand multiplication as repeated addition.
- ✔ Develop a better sense of numbers, especially to compose, decompose and factor integers.
- ✔ Understand and use the commutative and distributive properties of multiplication.
- ✔ Calculate area and perimeter of rectangles.
- ✔ Multiply two-digit numbers.
- ✔ Multiply binomials and trinomials (polynomials).
- ✔ Understand that two binomials are factors of a single trinomial.
- ✔ Combine like terms.

KEY VOCABULARY

rectangular array: an arrangement of items in rows and columns

area model: a type of rectangular array, with arranged squares in a grid

area: the size of a surface given in square units: 18 square inches

perimeter: the distance around a two-dimensional shape, given in linear units: 18 inches

factors: two or more numbers that can be multiplied together to get another number (the product): 1, 2, 3, 6, 9, and 18 are factors of 18

expression: a collection of numbers, symbols, and operators put together to show a value (doesn't include equal sign): 3×6

equation: a statement in which the equal sign indicates that two expressions have the same value: $3 \times 6 = 18$

commutative property of multiplication: the mathematical law stating that order doesn't matter in multiplication: $3 \times 6 = 6 \times 3$

distributive property: the sum of two numbers times a third number is equal to the sum of the products of the third number and each of the first two numbers: $3 \times (2 + 4) = 3 \times 2 + 3 \times 4$, or $a(b + c) = ab + ac$

monomial: an algebraic expression with one term: x, x^2, or $2x$

binomial: an algebraic expression with two terms: $2x + 3$

polynomial: an algebraic expression containing more than two terms: $x^2 + 2x + 1$

ACTIVITY 1 Using Arrays to Explore Multiplication

MATERIALS:

- Images of a carton of eggs, a six-pack of soda, and a 3 × 4 muffin tin
- 1-inch square tiles, or cut-out paper squares
- How Many Muffins Could You Make? (handout)
- Graph paper (ideally would have a grid on one side, with empty page on the other side)
- (optional) Multiplication Dot Array (handout)

STEPS:

1. Tell the class that you are going to show them some photos for only a second or two each and ask them to tell you how many items there are in each photo.

2. Flash these images quickly one at a time on a projector or on a printout. After each image, ask *How many? How did you know?*

   a. carton of eggs

   b. six-pack of soda

3. Show the image of a muffin pan (3 × 4) on a projector or on a printout. After each image: *How many? How did you know? How is this different from a carton of eggs or a six-pack of soda?* Some students might count each muffin. Show the photo just long enough to see that there are four groups of 3, so that students are encouraged to think in terms of groups.

4. Give students the handout How Many Muffins Could You Make? Ask them to think about the different ways they could express the quantity *12 muffins* mathematically. Possible responses include:

   - 3 + 3 + 3 + 3
   - 4 + 4 + 4
   - four groups of 3
   - three groups of 4
   - 4 × 3
   - 3 × 4

5. With each of these responses, it's useful to go back and ask students to demonstrate how these different "seeings" work. The muffins can be counted in a number of different ways and always end up being 12. You might ask a couple questions to make sure students notice this:

2

This is a way to tap into our students' intuitive understanding of groups. They will know at a glance that there are 12 eggs and 6 sodas. They will probably not know how many sections a muffin tin has, however. Some students will count four groups of 3 or three groups of 4. Other students will count one by one.

4

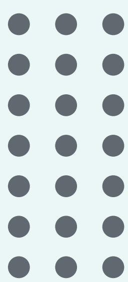
There are a few important concepts that can be connected to these responses:

- the connection between multiplication and addition—multiplication can represent groups of equal quantities (repeated addition)
- the commutative property (order doesn't matter in addition and multiplication), so 4 × 3 is the same as 3 × 4 give the same result

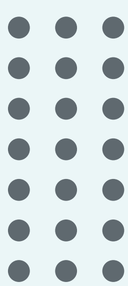
- *Oh, so if I add four groups of 3, that gives me the same as three groups of 4? Can someone explain how that works?*
- *4 × 3 is the same as 3 × 4? Does that always work? Is 5 × 4 the same as 4 × 5? 8 × 3 and 3 × 8? Can I reverse the order of any two numbers when multiplying and get the same answer? Why do you think this works?* Give students a few minutes to prove this to themselves.

TEACHER'S NOTE (EXTRA STUDENT SUPPORT)

With pre-HSE classes, you might spend more time working with dot arrays as a way for students to become comfortable moving from counting dots and adding groups to eventually multiplying to find the quantities. Create a worksheet with different dot arrays involving the multiplication you would like the class to practice. Draw dot arrays on the handout, *How Many Dots Are There?*, to represent different multiplication problems, which students can complete in this way:



______ **(total)** ______ **groups of** ______	**21** **(total)** **3** **groups of** **7**



______ **(addition)** ______ **(multiplication)**	**7+7+7** **(addition)** **3 x 7** **(multiplication)**

ACTIVITY 2 A Measured Area Model

MATERIALS:

• 1-inch square tiles (24 for every two students) • graph paper

We now move from counting objects like eggs, sodas, and muffins to counting more abstract quantities. However, we start in a very concrete way by having students use manipulatives to explore the concept of area.

STEPS:

1. **Put students into groups of two or three.** Give 24 one-inch tiles to each pair of students. Tell them that their task is to find how many different ways they can organize the 24 squares into a rectangle. Encourage students to play with the tiles and move them around in different configurations

2. **Next, hand out graph paper** and ask students to draw each of the four rectangles they found: 4 × 6, 3 × 8, 2 × 12, 1 × 24. Some students might argue that a 4 × 6 rectangle is different from a 6 × 4 rectangle. Take this opportunity to reemphasize the commutative property of multiplication. You can also take a drawing of a 4 × 6 rectangle and rotate the paper. Does this change the size of the rectangle?

You can find a detailed explanation of the transition from the dot array to the area model for multiplication on EngageNY: https://www.engageny.org/resource/multiplication-with-arrayarea-models-and-the-rekenrek

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Check out the Gold Rush Problem in the Reflective Teaching section for a problem that explores area and can be used to refer back to this activity later.

INTRODUCING AREA

Ask your students what they imagine when you say you want to measure the area of something. If students have ideas, they could include the following:

Length times width.	**Perimeter?**
Base times height.	**Covering something.**

Students who make one of the comments on the left are recalling formulas they memorized, possibly without really understanding. Students who say perimeter probably know that there is some relationship between area and perimeter, but might not be sure what the relationship is. A student who says something like “covering something” is on the right track. Have your students note the following definition of area: *size of a surface*. Examples of surfaces include the chalkboard, the tops of student desks/tables, and the walls. A surface can also include contours or bumps, such as a globe or a person’s head. A surface that can be measured using area is something you could imagine painting over. Ask volunteers to submit other real-world examples of area, and ask the class if they agree.

Area is measured in square units (square inches, square feet, square meters). When we measure area, we are essentially counting squares. Perimeter, on the other hand, is a measure of length and is measured in linear units (inches, feet, meters). Instead of counting squares, students should imagine measuring around their rectangles with a ruler or tape measure. If you wrapped a string around the rectangle, how long would it have to be?

We encourage teachers to help students begin to calculate area by physically counting squares. Kitchen tiles are a good example. Even better, if you have 12-inch square tiles in your classroom, point out to your students how you can measure the area of a section of the classroom floor by simply counting tiles. At this point, it is helpful for them to experience counting squares in order to remember the concept. If they just memorize the procedure of multiplying length times width, it's easy to confuse this with the procedure for finding perimeter.

At some point in the discussion, we hope that a student will remark that the area of any rectangle can be found by multiplying the measures of the two sides. We will want to be precise with this description, so ask some clarifying questions of your students. *Can I find the area by multiplying the lengths of opposite sides? Why doesn't that work?* After this clarification, your students have the ability to find the area of rectangles without a visible grid, and only with the lengths of adjacent sides. Most importantly, they know what the product of the lengths of a rectangle's adjacent sides represents: the number of squares that can cover its interior surface.

3. Tell your students to take a few minutes to determine the area and perimeter of each rectangle.

4. **Now ask students to describe the rectangles they made.** As volunteers describe each rectangle, draw the rectangle on the board, including the grid lines so that it is possible to count the number of squares in each rectangle. After students have shared their rectangles, ask if we have found all the possible rectangles with 24 squares. *Is it possible that we missed one? How can we be sure?*

5. Next ask, *What do you notice about the area and perimeter in these four rectangles?* Possible responses include the following, though you should welcome other observations from your students.

   - The area didn't change, but the perimeter did.
   - It's possible to get the same area with different widths and lengths.
   - 1 × 24 is one group of 24, 2 × 12 is two groups of 12, etc.
   - 1 × 24, 2 × 12, 3 x 8, and 4 x 6 are all equal to 24.
   - The bigger the difference between the two numbers, the larger the perimeter.
   - The closer the two numbers are in size, the smaller the perimeter.

6 Before moving on, students should recognize that all of the rectangles on the board have adjacent sides that are factors of 24. Helping students recognize this fact may require a line of questioning that connects the rectangles to factors explicitly.

- *What do all these rectangles have in common?* They all have 24 squares.
- *How do we know that they all have 24 squares?* We can count them.
- *Is there any other way we could prove they each have 24 squares?* We could multiply the length and width. 1×24, 2×12, 3×8, and 4×6 all equal 24.
- *So, the length and width of each of these rectangles (1, 24, 2, 12, 3, 8, 4 and 6) have a relationship with 24. Does anyone know the name for this relationship?* If no one does, tell the group that these numbers are factors of 24 and are usually written in increasing order: 1, 2, 3, 4, 6, 8, 12, 24.

7 If it hasn't come up, now is a good time to talk about factors. Amy Vickers uses the following questions with her students:

- What is a factor?
- How would you explain a factor in the context of rectangular arrays?
- What are a few examples of numbers that are not factors of 24? How would this look in a rectangular array?
- What equations can you write to describe the arrays that use multiplication?
- What equations can you write to describe the arrays that use division?
- What is the relationship between multiplication and division?
- For what other math topics is an understanding of factors essential?

8 Tell students that you are going to give them a series of multiplication problems that you would like them to show on their graph paper. Start with the following:

a. 9×4

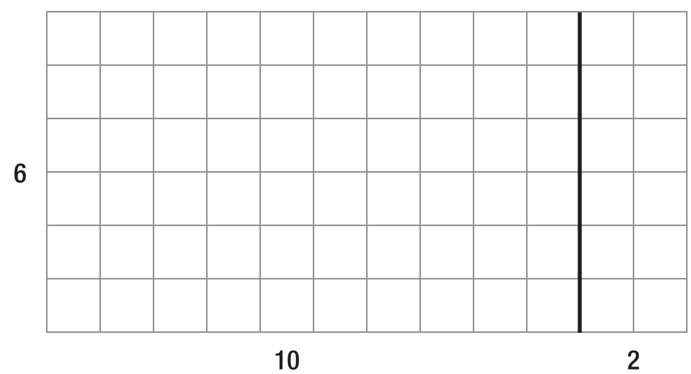
Ask students to draw this as a rectangle on their graph paper. They should label the length of the edges and write the area on the inside of the rectangle: *Area = 36 squares.*

b. 12×6

When students finish, ask for volunteers to explain how they came up with 72. Possible responses include:

- *Six groups of 12*
- *Twelve groups of 6*
- *Multiplied 12×6 or 6×12*
- *Multiplied 10×6 and added two groups of 6*

9 12×6 is a good opportunity to introduce the distributive property. Draw a 6×12 grid horizontally on the board and tell the class you want to explore the 12×6 further. Make a heavy line dividing the 10 from the 2, to divide the 12 in two pieces. Label the measurements of the edges.

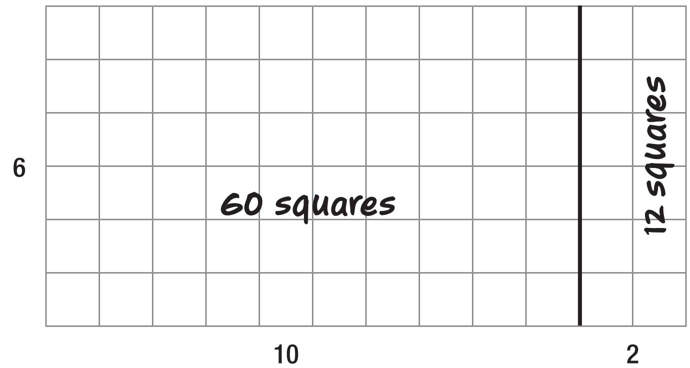


a. Ask, *Does breaking 12 into 10 and 2 change the total number of squares in the rectangle? How could you prove that we still have the same number of squares?* Possible answers could include:

- *Counting all the squares to see if there are 72*
- *Adding the 10 and the 2 to make 12, then multiplying by 6*

b. Ask, *What if we wanted to find the area of the 6 x 10 rectangle first? How many squares are in that part?* You might block the 2×6 part of the grid so that students can only see the 6×10 rectangle (six groups of ten, ten groups of six, 10×6, or 6×10). Write the total number of squares in the 6×10 section.

c. *And what about the right side? How many squares are there?* Block the 6×10 rectangle, so students can only see the 6×2 rectangle (two groups of six, six groups of two, 2×6, or 6×2). Write the total number of squares in the 6×2 section.



d. *So, how many total squares are there?* 60 + 12 = 72.

e. Rewrite the problem on the board as 6 (10 + 2) = 72. Ask students to explain how this equation connects to the area model drawn on the board.

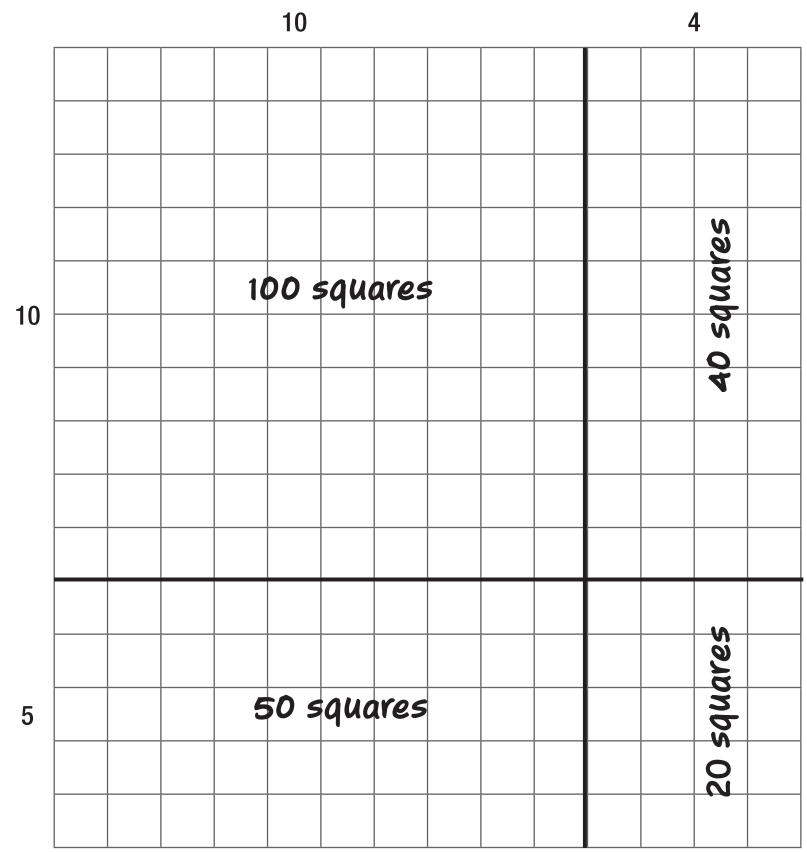
10 At this point, students should continue using area models to calculate the number of squares with some problems on their own. Say, *On your own, try using this model to multiply 15 × 9. After you draw your big rectangle, break the side that is 15 squares long into 10 and 5, then use the same method that we just did for 12 × 6.*

- 15 × 9
- 18 × 4
- 24 × 5
- 14 × 15

11 To process 14 × 15, guide students to break the 14 into 10 and 4, and the 15 into 10 and 5. Students should end up with a grid similar to the one below and then calculate the area of each of the four sections (4 × 5 = 20, 4 × 10 = 40, 10 × 5= 50, 10 × 10 = 100)

How could we figure out the total area?

Answer: Add up the total squares for each section (20 + 40 + 50 + 100 = 210).



11

This might be a good time to show students how the area model will give the same answer as the standard algorithm for multiplying two-digit numbers. If students are comfortable multiplying multi-digit numbers, they should continue to do multiplication in the way that works for them. The purpose of these activities is not to replace the way they do multiplication, but deepen our understanding of multiplication and connect it to other situations. Challenge: Can you use the area model to explain why the standard algorithm works?

ACTIVITY 3 The Distributive Property

MATERIALS:
- **Multiplying with Parentheses (handout)**

This is a great moment to teach the distributive property of multiplication, using the previous work as an example. This concrete example of distribution will help students remember that they need to multiply by each term within parentheses.

The handout Multiplying with Parentheses, can be used to explore the distributive property with students:

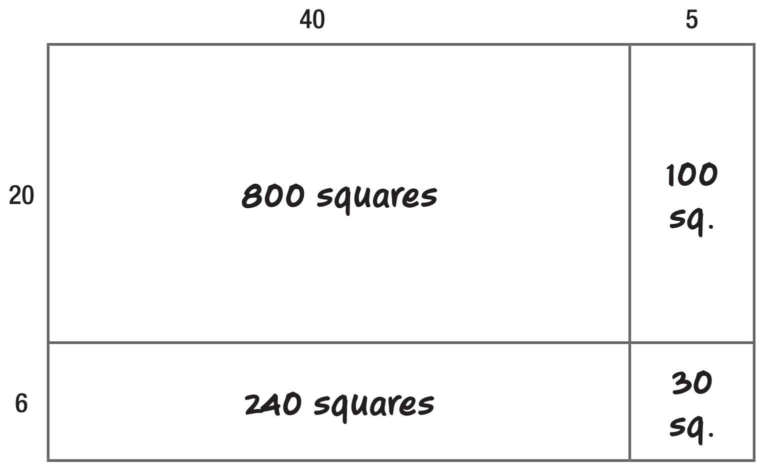
1. Say, *What does it mean to* simplify an expression? *Can someone explain the steps that Paul took when he simplified 5 (3 + 12)? What do you think of what he did? Would you do it differently or the same way? What is Paul's confusion?*
2. Ask a student to draw an area model of the expression on the board.
3. Ask, *How can the area model help us understand 5 (3 + 12)?*
4. Students should be able to connect their explanations to the drawing. Some students may previously have known the distributive property, but they should be able to explain why calculating the area of both rectangles is necessary to find the total squares.

ACTIVITY 4 An Abstract Area Model

After students have practiced and become comfortable using the concrete method of counting squares using grid paper, we can move to an abstract area model by leaving the grid paper behind.

1. Give the class a multiplication problem whose area model won't fit on their graph paper, like 45 × 26. You might count the number of rows or columns on the graph paper you give them and then choose a number that is just a little bit too big. Students will try to count out the rows and columns. When students realize that they aren't able to able to use the grids, tell them to turn the paper over and do it on the back.

2. Let your students know that they don't have to draw all the lines in the grid. They can just draw a rectangle. Ask your students to use the same technique for breaking up the numbers and drawing lines to divide the rectangle. Here's one way a student might break up the rectangle and calculate the area for each section.



3 Next to the area model, show the calculations for the area of each of the smaller rectangles, as well as the total area:

$6 \times 5 = 30$

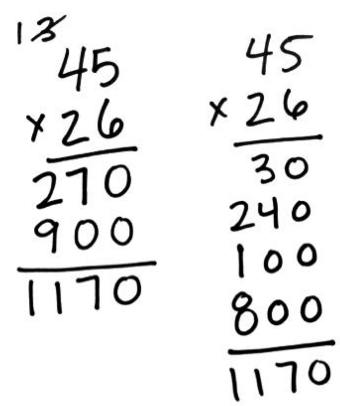
$6 \times 40 = 240$

$20 \times 5 = 100$

$20 \times 40 = 800$

total area = 1170 squares

4 You can use this opportunity to make connections to the standard way of doing multiplication in the United States. Ask students to look at the following two models and make connections to the area model. The image to the right shows the standard algorithm on the left and the partial product method on the right. Students will probably be familiar with the standard algorithm, though students often struggle to record products with the proper place value. The partial product method separates the calculations and retains the place value for each. The second line, for example, shows the product of 6 and 40, rather than multiplying 6 and 4 as part of the multiplication of 45 times 6.



What do your students notice about how the totals are calculated and arranged? Where do they see any connections to the area model of multiplication? Which of these procedures for multiplication do they prefer? Does anyone think they might use the partial product method in the future?

4

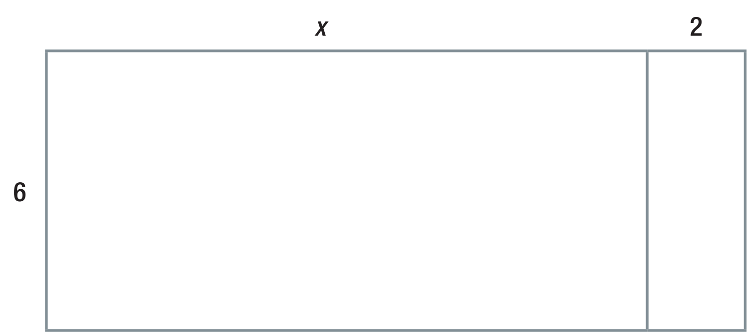
A clear demonstration of the partial product algorithm can be found at http://everydaymath.uchicago.edu/teaching-topics/computation/mult-part-prod/.

ACTIVITY 5 Applying the Area Model to the Multiplication of Polynomials

In order to start working with binomials and polynomials, it is important for students to understand and be able to use the distributive property. The area model gives students a way to organize their work. When they move on to college, we will want them to be able to use FOIL (**F**irst, **O**utside, **I**nside, **L**ast) as well, but starting with the area model connects an abstract idea to a concrete model that students can hold on to.
We continue by substituting variables into the area models we have already worked on so that students see the continuity in the approach. We follow the same logic as before, but now we apply it to expressions that include variables.

1. As a review, draw an abstract area model of 6(10 + 2). Students should be able to say that the total number of squares is $(6 \times 10) + (6 \times 2)$.

2. To the right, draw an area model of $6(x + 2)$.



If students are unfamiliar with variables, you can say that x just means we don't know the width of the rectangle on the left. Even though we don't know how wide it is, we do know that 6 times that width would tell us how many squares there are in the left part of the rectangle. If x was 2, then there would be 12 squares. If x was 3, there would be 18 squares. If x was 100, there would be 600 squares. x could be anything.

a. Ask students what the area of the rectangle on the right would be. *Answer:* 12 squares.

And the area of the rectangle on the left? *Answer:* 6 times x, or $6(x)$, or $6x$ squares. $6x$ is a way of saying that even though we don't know how many squares are there, we do know that there will be 6 times whatever x is.)

Write $6x$ and 12 inside the rectangles.

b. Below the area model, write $6(x + 2) = 6x + 12$.

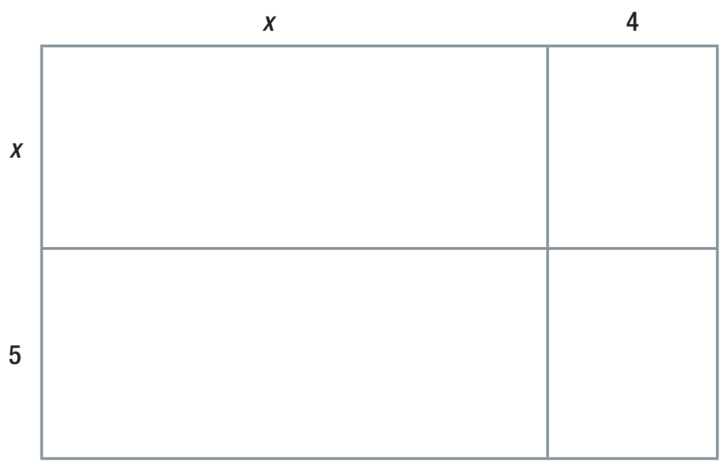
c. Following the model on the board, students should practice multiplying the pairs below, drawing area models and writing expressions for their answers:

(1) 9 and $(x + 5)$

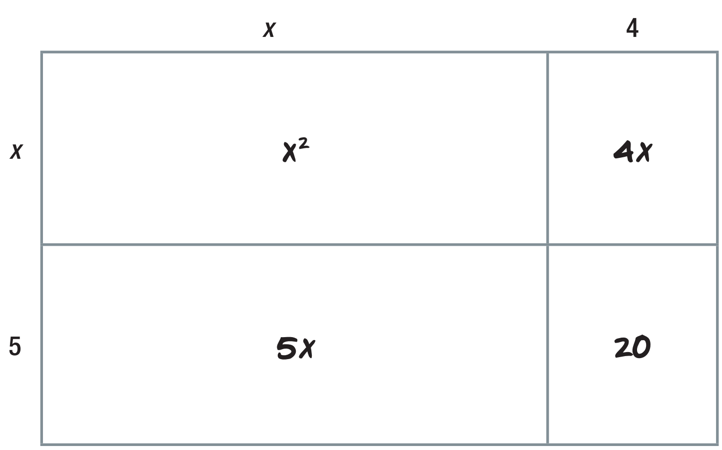
(2) 4 and $(x + 8)$

(3) 5 and $(2x + 4)$

3 For a second example, tell the class to draw an area model of $(x + 5)(x + 4)$. Their model should look like this:



a. Ask students to help you fill in the area of each rectangle, starting with the bottom right one ($4 \times 5 = 20$ squares). Now, move to the top right and bottom left rectangles. Students should be able to fill in the area of these rectangles with $4x$ and $5x$, respectively. The top-left box may present a challenge for students who haven't studied square numbers or exponents. At the beginning, I would accept the following answers: x times x, or x times itself, $x \times x$, and, eventually, x^2.



3a

This is a good opportunity to talk to students about how x is not generally used to mean multiplication in algebra. Students should start to recognize multiplication in the following forms:

$$5 \cdot 6 = 30$$

$$5(6) = 30$$

In this case, we want to express that x is multiplied by x. Do students see how using x to represent multiplication might cause a problem?

3b

You can define term as either a single number or variable, or numbers and variables multiplied together. The areas x^2, $4x$, $5x$, and 20 are all examples of terms.

b. Below the area model, write $(x + 5)(x + 4) = x^2 + 4x + 5x + 20$.

i. Ask the class is there is anything you can do to simplify the equation, or write it with fewer terms. If no one brings up combining the **terms** $4x$ and $5x$, explain that, in algebra, we can combine items when the variable parts are the same *(combining like terms)*. Ask the class to point out which two terms have the same variable part. These two terms can be combined. If we have 4 x's and 5 more x's, then we must have 9 x's altogether.

ii. Write $(x + 5)(x + 4) = x^2 + 9x + 20$ below the first equation.

iii. Prompt the class to look at the expression $x^2 + 9x + 20$ and think about how it connects to the area model. Here are some questions you can ask:

- *Where does the x^2 come from?*
- *Where does the 9x come from?*
- *Where does the 20 come from?*

Here, we want students to see the connection between the product and sum of 4 and 5, as they appear in the polynomial that is the product of the two binomials.

- *Do you see any connections between 9 and 20? What is interesting about these two numbers?*
- *Look at the area model. How did we get 9? 4 + 5. How did we get 20? 4 × 5. Hmm… What do you think about that?*
- *So, 4 + 5 is 9 and 4 × 5 is 20. That's interesting. Do you think that would happen if we multiplied other quantities? Let's do some other ones and see if it happens again.*

4 Give students practice multiplying binomials. Some samples you could use are:

$(x + 1)$ and $(x + 2)$

$(x + 3)$ and $(x + 4)$

$(x + 2)$ and $(x + 5)$

a. At this point, you could come back to the connection between the x term and the constant. Ask students to look at the trinomial expressions that were products of the binomials they multiplied. Ask, *What do you notice about the x term and the constant?* If there are students who don't see the connection, you might do a quick pair/share and ask students to discuss the following example:

$$(x + 3)(x + 4) = x^2 + 7x + 12$$

b. Then put this example on the board:

$$(x + 2)(x + 5) = x^2 + 7x + 10$$

Ask for volunteers to talk about what they notice when they compare these two examples. When students have confirmed that the constants from the two binomials are added to get the x term in the trinomial and multiplied to get the constant in the trinomial, you might give them the attached diamond pattern worksheets that will prepare them for factoring polynomials.

4a

You can define the constant as a term in an algebraic expression that has a value that doesn't change, because it doesn't have a variable.

MAKING YOUR OWN NUMBER DIAMOND WARM-UP (OPTIONAL)

As you create your own number diamond problems, rotate the given information so that students have to adjust their reasoning. Include decimal examples, particularly those that helps students practice their mental math strategies, and that include the benchmark decimals (0.5, 1.5, 2.5, etc.). Be careful about items that only include the product and sum—these can be very difficult when decimals are involved. Have fun!

How Many Muffins Could You Make?

How many different ways can you express the number of muffins that this pan holds?

Multiplication Dot Array

How Many Dots Are There?

____
(total)
_____ groups of _____

____
(addition)

____
(multiplication)

____
(total)
_____ groups of _____

____
(addition)

____
(multiplication)

____
(total)
_____ groups of _____

____
(addition)

____
(multiplication)

____
(total)
_____ groups of _____

____
(addition)

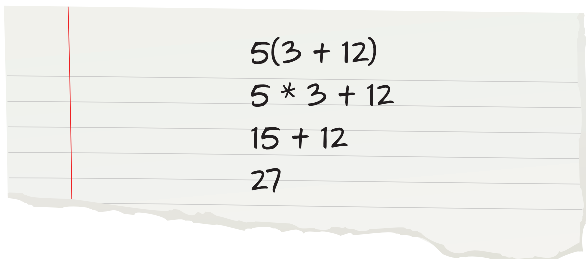
____
(multiplication)

Multiplying with Parentheses

Paul's teacher wrote the following expression on the board for the class to simplify:

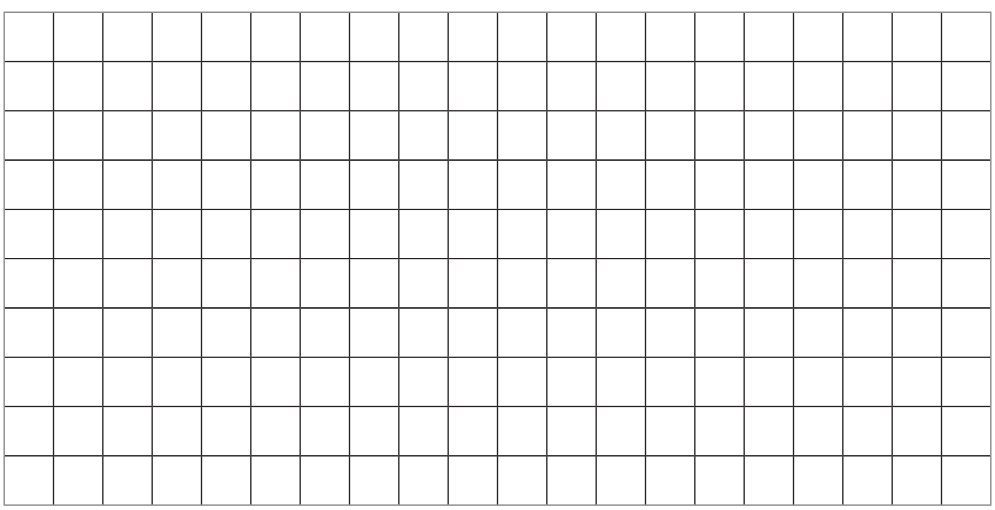
$$5(3+12)$$

Here is an excerpt from Paul's notebook.



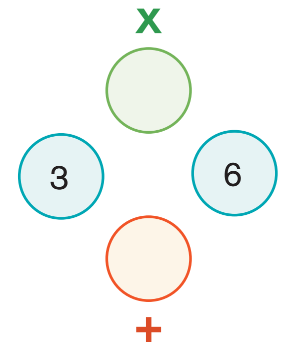
What do you think of Paul's work? Explain your thinking.

Draw an area model of 5(3+12) below:



How many squares are there?

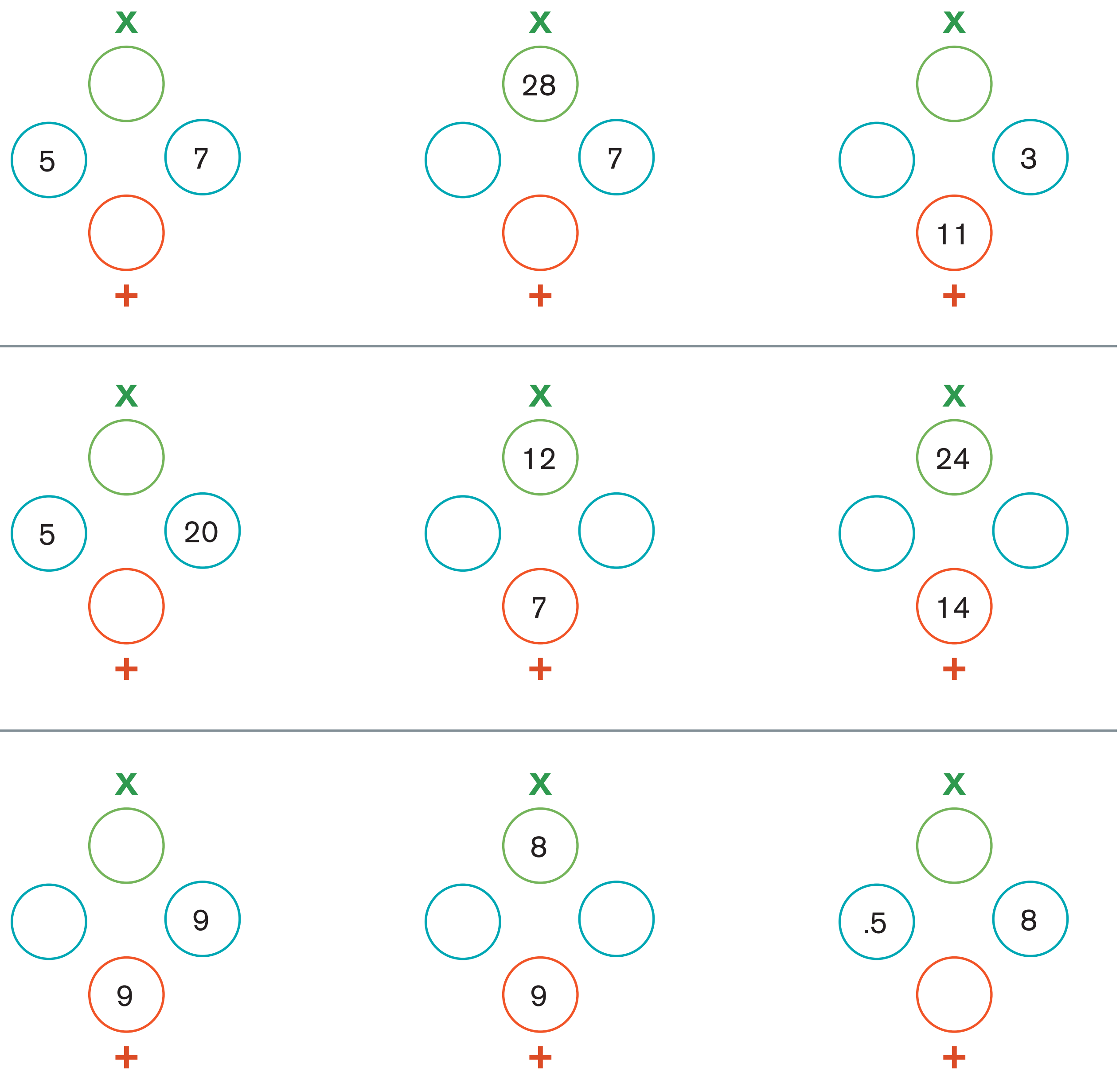
Number Diamond 1



A Number Diamond always includes four circles. The two horizontal circles are our two base numbers. The top circle is for the product of the two base numbers. The bottom circle is for the sum of the two base numbers.

In each Number Diamond, you will be given two numbers, and you will have to figure out the missing two. In this example, you are given the two base numbers. What is the product of 3 and 6? What is the sum of 3 and 6?

Fill in the missing circles for each Number Diamond.

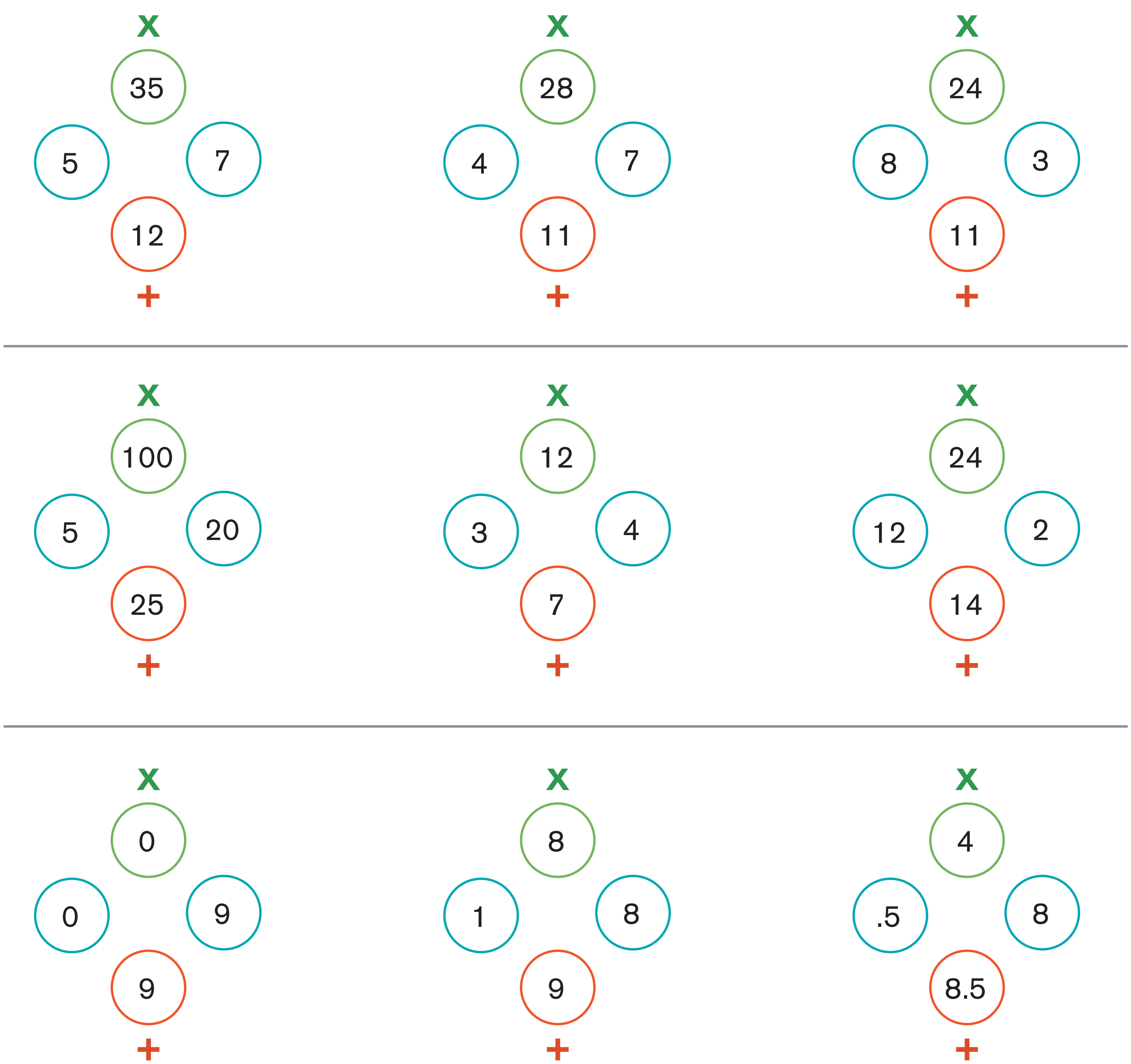


Source: Adapted from *Foundations for Algebra* by College Preparatory Mathematics

Solution: Number Diamond 1

These number puzzles provide a rich and engaging opportunity for students to improve their arithmetic skills, including whole numbers, decimals, fractions, and even negative numbers.

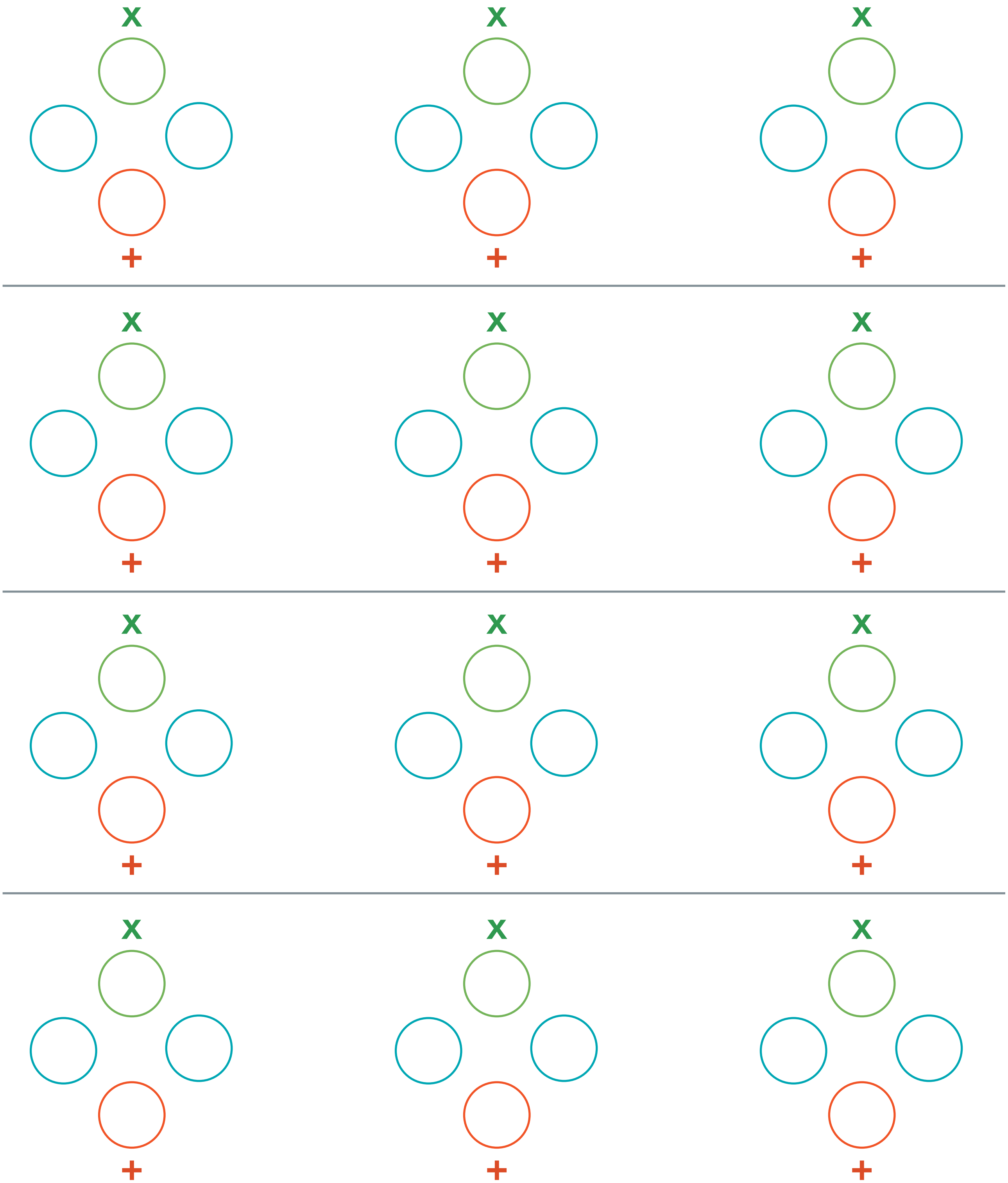
These puzzles also foreshadow skills students will need if they go on to college-level mathematics. This particular type of factoring requires the kind of reasoning called for when students need to figure out the base numbers when given their product and sum.



Source: Adapted from *Foundations for Algebra* by College Preparatory Mathematics

Number Diamond

Fill in the missing circles for each Number Diamond.



Source: Adapted from *Foundations for Algebra* by College Preparatory Mathematics